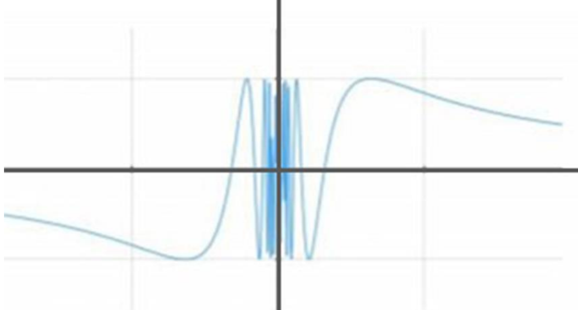
Teaching Limits: A Guide for Calculus Instructors



Maria Luisa Torres, Ph.D.

BIM Because It is Math

Edition 1

Teaching Limits:

A Guide for Calculus Instructors

Author: Maria Luisa Torres, Ph.D.

BIM **Because It is Math**

Edition 1

Contents

List of Figures

Introduction

Sadly, Calculus has become an insuperable obstacle for some students interested in pursuing careers in sciences, engineering, economics, social sciences and others.

Calculus is not a difficult subject, it has been made difficult because differentiation and integration are limits, and when limits are not understood calculus does not make any sense.

Instructors insist on teaching calculus without teaching the definitions of limits, insisting on using intuition instead. Mathematics is about solving problems, problems are solved with knowledge followed by intuition not the other way around.

Is it possible to teach the fundamental theorem of arithmetic without knowing the definition of a prime number? Do we dare to ask students to solve area problems without knowing what an area is? Yet, we insist on teaching calculus without teaching the definitions of limits. We insist on navigating a canoe without a paddle.

After so many generations of teaching limits without introducing their definitions as a starting point, it is difficult to convince instructors to do it differently, particularly when most calculus textbooks follow this approach. This *guide* intends to help calculus instructors on this regard.

Chapters 1 to 5 cover definitions and results about limits as they should be learned: starting with definitions from which proved results are built upon. This material is what any calculus instructor should know. We present a rigorous approach so that definitions and proved results, i.e. theorems, proportions, etc., are understood. It is not a good idea to skip them. Instructors must only teach material they master.

Chapters 6 to 10 guide instructors on how to teach limits omitting their definitions or making them optional, as it is customary in most calculus courses.

In most calculus courses, proofs of theorems, propositions, lemmas, and corollaries may not be required (ideally they should). If they are not, at the very least they must be understood and applied correctly.

In Appendix A we present the graphic representations of all the definitions of limits studied in this *guide*.

In Appendix B we present complete solutions of all the exercises of this *guide*, to make clear what is expected from instructors and students. Yet, we do not recommend to provide complete solutions to all exercises assigned to students, as it is customary in high school math courses. This is not desirable because students must learn to check their own work and eventually the work of others. Students must developed confidence in their knowledge and stop depending on someone to tell them whether their work is correct or not.

As instructors, our aim is to teach our students the necessary skills in order for them to become independent thinkers. We want them to be self-confident on their learning. They will never be, if we constantly and systematically provide them with the solutions of every single exercise assigned to them. Moreover, students are quite capable of finding their own correct answers, maybe different than their instructors' and this is a very good thing. Do not be afraid to let your students struggle, they will be rewarded with long lasting knowledge.

We ask you to please avoid, at all cost, the bad habits of

- using "tables of values" to evaluate limits,
- giving yourself free licensing in the use of the symbol ∞ with arithmetic operations,
- not distinguish between infinite limits, limits which do not exist, and undefined limits, and
- insisting on given "meaning" to the division by zero when evaluating limits.

These bad habits have been preventing many generations of students from learning, understanding and applying limits to solve problems.

All this being said, let us insist that not knowing what limits are (i.e. their definitions), it is not knowing calculus. We cannot leave students in the dark and insist that it is not important or it does not matter if they do not know what a limit is. Mathematician are not the only ones who must know how to apply limits to solve problems.

Students are capable of mastering limits if they are taught properly. It is not true that the definition of a limit is too difficult a concept for students to understand.

In this *guide*, we give negations of definitions, examples and counterexamples to justify the validity or falsehood of what we learn. Students must know how to negate definitions and statements, because they must be able to determine what is and what is not.

Objectives

Students need to know what they are expected to master, and consequently on what they are being tested on. We state the objectives of the learning material before each set of exercises to illustrate their propose.

Suggested Exercises

Mathematics is learned, as everything, by doing. Drilling exercises are necessary to master algorithms and processes. But we must also offer exercises which challenge the creativity of our minds. We must learn to accept and attack challenges with excitement and interest for learning. In other words, we must enjoy the mastery of the subject.

A. **Give the negation of...** Students must know what is and what is *not*. They must know how to negate a definition to know what is not. The negation of a statement may not be trivial; it may require careful thinking. It is an enriching process which takes students to the truly understanding of mathematical statements.

B. **Give an example of....** A concept is understood when we can give an example of what is and what is not. This follows from the learning on how to negate statements. By giving examples students learn to create and justify their ideas.

C. **Give a function/graph which satisfies all the conditions listed below.** These type of exercises are a variation of “giving examples.” Hence, they engage students to clearly understand concepts.

D. **Learning from mistakes.** Students must become independent thinkers. They should be able to identify errors and correct them. There must be a time when nobody holds their hand.

E. **Explain.** To attest students’ understanding, it is necessary to ask them to explain what they are doing and why.

F. **Hints before solutions.** Hints encourage students to keep thinking, learning is about keep trying and never giving up.

We will make constant use of the rules of inequalities, the definition of the absolute value and its related inequalities as listed below.

Rules of Inequalities

1. If $a < b$, then $a + c < b + c$ for any c.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $b < c$, then $a < c$.
4. If $a < b$ and $c > 0$, then $ac < bc$.
5. If $a < b$ and $c < 0$, then $ac > bc$.
6. If $0 < a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.

Definition 0.1. The *absolute value* of any number a is the zero or positive number $|a|$ defined as

$$|a| = \begin{cases} a & \text{if} \quad a > 0 \\ -a & \text{if} \quad a < 0 \\ 0 & \text{if} \quad a = 0 \end{cases}$$

The inequalities listed below follow from Definition 0.1.

Inequalities

1. The distance between a and b is the same as the distance between b and a.

   $$|a - b| = |b - a|.$$

2. Triangle inequality.

   $$|a \pm b| \leq |a| + |b|.$$

3. $|a| - |b| \leq |a - b|$ follows from numbers 1 and 2.
4. $|a - b| < c$ if and only if $-c < a - b < c$.
5. $|a - b| > c$ if and only if $a - b < -c$ or $a - b > c$.

6. $-|a| \le a \le |a|$.

This *guide* is intended to be read primary as a PDF document. However, the page numbers of all references are provided for the benefit of those who wish to read it in printed form.

The author appreciates and welcomes corrections or recommendations at malutoga55@gmail.com.

Chapter 1
Finite Limits at a Number

In most calculus courses the definitions of limits are left as "optional" or out completely; they are considered to be "out of scope." Instructors are left with the task of teaching limits without defining them. As a consequence, difficulties arise and misunderstandings abound. Naturally, this is so because instructors approach the learning of limits "intuitively." But, intuition takes us only so far. Eventually, we reach a moment when we must *know* the definition of what we are working with.

A Number Close to Another

Once the meaning of "a number x is close to a number a" is understood, the definitions of limits start to sink in, and everything else falls into place.

We must to understand what we mean by "a number x is close to number a from the right, or left" on the real line.

Definition 1.1. A number x is *at the right* of a number a if

$$x - a > 0 \quad \Leftrightarrow \quad a - x < 0,$$

and it is *at the left* of a if

$$a - x > 0 \quad \Leftrightarrow \quad x - a < 0.$$

For example, the number π is at the right of the number 3 because $\pi - 3 > 0$ and the number e is at the left of the number 3 because $3 - e > 0$.

Two numbers are close if their *distance* is small. Hence, the need to establish mathematically, the *distance* between two numbers.

Figure 1.1 shows the distance between the number x and the number a on the real line.

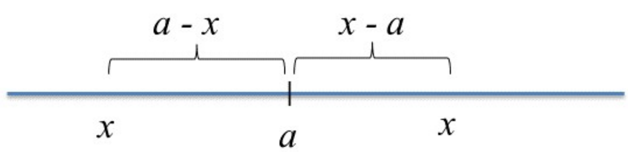
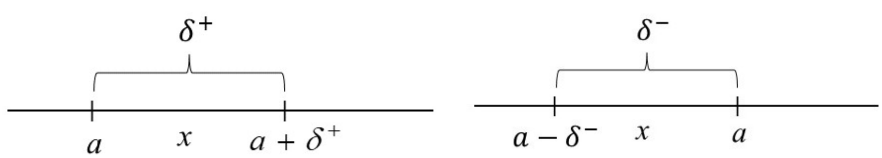


Figure 1.1: Distance between the numbers x and a on the real line

Figure 1.2 illustrates numbers x close to a number a from the right and left.



(a) x at the right of a. **(b)** x at the left of a.

Figure 1.2: Distance between x and a is less than δ.

If x is at the right of a, then $x - a > 0$ is the distance between x and a.
If x is at the left of a, then $a - x > 0$ is the distance between x and a.

On the other hand, by the definition of absolute value (page xii)

$$|x-a| = \begin{cases} x-a & \text{if} \quad x-a > 0 \\ a-x & \text{if} \quad a-x > 0 \\ 0 & \text{if} \quad x = a \end{cases}$$

Therefore, $|x - a|$ is the distance between any two numbers x and a.

If $\delta > 0$ is a positive number, then

$|x-a| < \delta$ indicates that the distance between the numbers x and a is less than δ.

$|x-a| = \delta$ indicates that the distance between the numbers x and a is equal to δ.

Thus, the numbers x and a are very close, if the positive number δ is very small.

Certainly, small and very small may mean different things, but what we want to convey is the "closeness" of two numbers.

Definition 1.2. a. A number x is *very close to a number a from the right* if there is a very small positive number $\delta^+ > 0$ such that

$$0 < x - a < \delta^+ \quad \Leftrightarrow \quad a < x < a + \delta^+.$$

We write $x \to a^+$ to denote the set of all numbers x very close to a number a from the right. That is,

$$x \to a^+ = \{x \in \mathbb{R} \mid 0 < x - a < \delta^+ \quad \text{for some} \quad \delta^+ > 0\}.$$

b. A number x is *very close to a number a from the left* if there is a very small positive number $\delta^- > 0$ such that

$$0 < a - x < \delta^- \quad \Leftrightarrow \quad a - \delta^- < x < a.$$

We write $x \to a^-$ to denote the set of all numbers x very close to a number a from the left. That is,

$$x \to a^- = \{x \in \mathbb{R} \mid 0 < a - x < \delta^- \quad \text{for some} \quad \delta^- > 0\}.$$

c. A number x is *very close to a number a from either right or left*, if there is a very small positive number $\delta > 0$ such that

$$0 < |x - a| < \delta \quad \Leftrightarrow \quad a - \delta < x < a + \delta.$$

We write $x \to a$ to denote the set of all numbers x very close to a number a from both right and left. That is,

$$x \to a = \{x \in \mathbb{R} \mid 0 < |x - a| < \delta \quad \text{for some} \quad \delta > 0\}.$$

Remark 1.3. *1. Observe that the number a does not belong to any of the sets $x \to a^+, x \to a^-$, nor $x \to a$.*

2. *In interval notation the set $x \to a^+$ is the open interval $(a, a + \delta^+)$, the set $x \to a^-$ is the open interval $(a - \delta^-, a)$, and the set $x \to a$ is the union of the open intervals $(a - \delta, a)$ and $(a, a + \delta)$, namely $(a - \delta, a) \cup (a, a + \delta)$.*

3. *The sets $x \to a^+, x \to a^-$, and $x \to a$ are nonempty because the set of rational numbers is dense in the real line.* [1] *In particular,*

   for every $a \in \mathbb{R}$ and every positive number $\delta > 0$, there is a rational number x, such that $a < x < a + \delta$.

[1] Instructors may wish to investigate this further by learning about the closure of the set of rational numbers in the real line.

Hence, the open interval $(a, a+\delta)$ is nonempty. Moreover, since $\delta > 0$ is arbitrary, there are infinitely many numbers x very close to a from the right. By the same argument, there are infinitely many numbers close to a from the left.[2]

Our interpretation of Definition 1.2 is that the set $x \to a^+$ consists of all numbers very close to a from the right. Similarly, the set $x \to a^-$ consists of all numbers very close to a form the left, and the set $x \to a$ consist of all numbers very close to a from the right and left.

We introduce the next definition, to establish what it means for a function f to have a property P, such as, defined, positive, negative, non zero, bounded, or continuous, close to a number.

Definition 1.4. a. A function f has the *property P on the right of a* if there is a positive number $\delta^+ > 0$ such that

$$f(x) \quad \text{has the property } P \text{ for every}$$

$$0 < x - a < \delta^+ \quad \Leftrightarrow \quad a < x < a + \delta^+.$$

We write

f has the property P for $x \to a^+$.

We read

f has the property P for every number very close to a from the right.

b. A function f has the *property P for x on the left of a* if there is a positive number $\delta^- > 0$ such that

$$f(x) \quad \text{has the property } P \text{ for every}$$

$$0 < a - x < \delta^- \quad \Leftrightarrow \quad a - \delta^- < x < a.$$

We write

f has the property P for $x \to a^-$.

We read

f has the property P for every number very close to a from the left.

[2] Instructors may want to prove these two claims.

c. A function f has the *property* P *around* a if there is a positive number $\delta > 0$ such that

$$f(x) \quad \text{has the property } P \text{ for every}$$

$$0 < |x - a| < \delta \quad \Leftrightarrow \quad a - \delta < x < a \text{ or } a < x < \delta + a.$$

We write

f has the property P for $x \to a$.

We read

f has the property P for every number very close to a from the right and left.

In the next example the property P is *positive*.

Example 1.1. a. Consider the segment of the graph of the sine function on the open interval $\left(0, \dfrac{\pi}{2}\right)$.

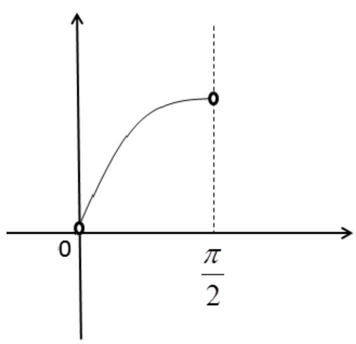


Figure 1.3: Graph of the sine function on the interval $\left(0, \dfrac{\pi}{2}\right)$

Figure 1.3 shows that

$$\sin x > 0 \quad \text{for every} \quad 0 < x < \frac{\pi}{2}.$$

Hence, parts (a) and (b) of Definition 1.4 (page 4) hold with the same positive numbers $\delta^+ = \delta^- = \dfrac{\pi}{2}$. Thus,

$$\sin x > 0 \quad \text{for} \quad x \to 0^+ \quad \text{and} \quad x \to \frac{\pi}{2}^-.$$

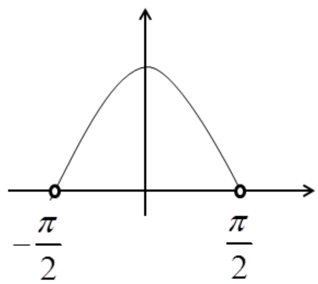


Figure 1.4: Graph of the cosine function on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. Similarly, consider the segment of the graph of the cosine function on the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Figure 1.4 shows that

$$\cos x > 0 \quad \text{for every} \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Hence, part (c) of Definition 1.4 (page 4) holds with the positive number $\delta = \frac{\pi}{2}$, and

$$\cos x > 0 \quad \text{for} \quad x \to 0.$$

Parts (a) and (b) of Definition 1.4 hold with the same positive number δ, and

$$\cos x > 0 \quad \text{for} \quad x \to \frac{\pi}{2}^- \quad \text{and} \quad x \to \frac{\pi}{2}^+$$

□

Identify the property P in the next two examples.

Example 1.2. For the positive number $\delta = 1$ we have that

$$\ln x < 0 \quad \text{for every} \quad 0 < x < 1.$$

Hence, by part (a) of Definition 1.4 (page 4) the function $\ln x$ is negative for $x \to 0^+$.

For the positive number $\delta = \frac{1}{2}$ we have that

$$\ln(1-x) \quad \text{is defined for every} \quad 0 < 1 - x < \frac{1}{2}.$$

Hence, part (b) of Definition 1.4 (page 4) the function $\ln(1-x)$ is defined for $x \to 1^-$.

Draw the graphs of $\ln x$ and $\ln(1-x)$ and convince yourself of these two claims. □

Example 1.3. Let x be a number between 2 and 3; thus,

$$2 < x < 3 \quad \Leftrightarrow \quad 0 < x - 2 < 1,$$

and for this x the function

$$f(x) = \sqrt{x-2} \quad \text{is defined.}$$

Hence, part (a) of Definition 1.4 (page 4) holds with the positive number $\delta = 1$, and $f(x)$ is defined for $x \to 2^+$. □

If a function f has a property P around a number a, then it has the property P on the right and left of the number a. This obvious truth is proved in the next proposition.

Proposition 1.5. *A function f has the property P around a number a if and only if it has the property P on the right and left of the number a.*

Proof. $\Rightarrow$) If $0 < |x - a| < \delta$ for some positive number $\delta > 0$, then $x \neq a$ and

$$-\delta + a < x < \delta + a \Rightarrow 0 < x - a < \delta \quad \text{and} \quad 0 < a - x < \delta.$$

Hence, if a function f has the property P around a number a, then it has the property P on the left and right of a with $\delta^+ = \delta = \delta^-$.

$\Leftarrow$) If the function $f(x)$ has the property P for all numbers x so that

$$a < x < a + \delta^+ \quad \text{and} \quad a - \delta^- < x < a$$

for some positive numbers $\delta^+, \delta^- > 0$, then for the positive number $\delta = \min(\delta^+, \delta^-) > 0$

$$-\delta < x - a < \delta \Rightarrow -\delta^- \leq -\delta < x - a < \delta \leq \delta^+.$$

Hence, $f(x)$ has the property P for $0 < |x - a| < \delta$.
Therefore, the function f has the property P around a.

Q.E.D.

We learn and understand definitions because it is important to know what a concept is. It is equally important to learn to negate definitions to know precisely what it is not.

Negation of Definition 1.4 (page 4).

a. A function f does not have the property P at the right of a if for any positive number $\delta^+ > 0$

$$f(x) \quad \text{does not have the property } P \text{ for some}$$

$$0 < x - a < \delta^+ \quad \Leftrightarrow \quad a < x < a + \delta^+.$$

We write

$$f \text{ does not have the property } P \text{ for } x \to a^+.$$

b. A function f does not have the property P at the left of a if for any positive number $\delta^- > 0$

$$f(x) \quad \text{does not have the property } P \text{ for some}$$

$$0 < a - x < \delta^- \quad \Leftrightarrow \quad a - \delta^- < x < a.$$

We write

$$f \text{ does not have the property } P \text{ for } x \to a^-.$$

c. A function f does not have the property P around a if for any positive number $\delta > 0$

$$f(x) \quad \text{does not have the property } P \text{ for some}$$

$$0 < |x - a| < \delta \quad \Leftrightarrow \quad a - \delta < x < a \;\text{ or }\; a < x < \delta + a.$$

We write

$$f \text{ does not have the property } P \text{ for } x \to a.$$

Negation of Proposition 1.5 on page 7.

A function f does not have the property P if and only if either

i. f does not have the property P for $x \to a^+$, or

ii. f does not have the property P for $x \to a^-$.

See how we apply these negations in the next three examples.

Example 1.4. The function

$$\sin\left(\frac{1}{x}\right) \qquad \text{is neither positive nor negative for } x \to 0.$$

For any $\delta > 0$, there is a positive integer k such that

$$k > \frac{1}{2\delta\pi} > 0 \quad \Rightarrow \quad 0 < u = \frac{1}{2k\pi} < \delta.$$

For this u we have

$$\sin\left(\frac{1}{u}\right) = \sin(2k\pi) = 0.$$

By part (a) of the negation of Definition 1.4 on page 8

$$\sin\left(\frac{1}{x}\right) \qquad \text{is neither positive nor negative for } x \to 0^+.$$

The claim follows from part (c) of the negation of Definition 1.4. □

Example 1.5. The function

$$\csc\left(\frac{1}{x}\right) \qquad \text{is not defined around } 0.$$

Indeed, by Example 1.4, for any positive number $\delta > 0$ there is a number u so that

$$0 < u < \delta \qquad \text{and} \qquad \sin\left(\frac{1}{u}\right) = 0.$$

Hence,

$$\csc\left(\frac{1}{x}\right) \text{ is undefined.}$$

Hence, by part (a) of the negation of Definition 1.4 (page 8),

$$\csc\left(\frac{1}{u}\right) \qquad \text{is undefined for } x \to 0^+.$$

The claim follows from the negation of Proposition 1.5 (page 8). □

Example 1.6. The function

$$f(x) = \sqrt{x-1}$$

whose graph is shown in Figure 1.5, is not defined for $x \to 1^-$.

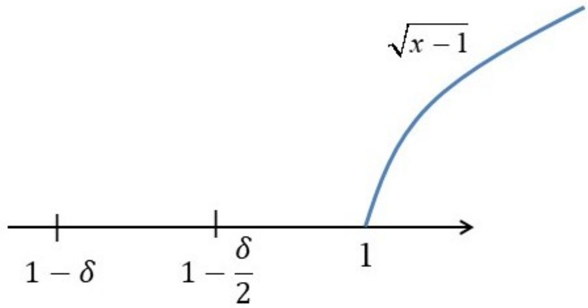


Figure 1.5: Graph of $f(x) = \sqrt{x-1}$ on $(1-\delta, \infty)$

Indeed, for any positive number $\delta > 0$ the number

- $x = 1 - \dfrac{\delta}{2}$ is such that $0 < 1 - x = 1 - 1 + \dfrac{\delta}{2} = \dfrac{\delta}{2} < \delta$, and
- $-\delta < x - 1 < 0 \quad \Rightarrow \quad 1 - \delta < x < 1.$

Thus, the number x is on the left of 1 and $f(x)$ is not defined for this particular x. Hence, by part (b) of the negation of Definition 1.4 (page 8).

$$\sqrt{x-1} \quad \text{is not defined for} \quad x \to 1^-.$$

□

The next definition is a particular case of Definition 1.4 (page 4), where the property P is *defined*.

Definition 1.6. a. A function $f(x)$ is *defined at a from the right*, if there is a positive number $\mu_+ > 0$ such that

$$f(x) \quad \text{is defined for every}$$

$$0 < x - a < \mu^+ \quad \Leftrightarrow \quad a < x < a + \mu^+.$$

It is represented by

$$f(x) \quad \text{is defined for } x \to a^+.$$

b. A function $f(x)$ is *defined at a from the left*, if there is a positive number $\mu_- > 0$ such that

$$f(x) \quad \text{is defined for every}$$

$$0 < a - x < \mu_- \quad \Leftrightarrow \quad a - \mu_- < x < a.$$

It is represented by

$$f(x) \quad \text{is defined for } x \to a^-.$$

c. A function $f(x)$ is *defined around* a, if there is a positive number $\mu > 0$ such that

$$f(x) \quad \text{is defined for every}$$

$$0 < |x - a| < \mu \quad \Leftrightarrow \quad a - \mu < x < a \;\text{ or }\; a < x < \mu + a.$$

It is represented by

$$f(x) \quad \text{is defined for } x \to a.$$

Remark 1.7. *From Definition 1.6 (page 10) and Remark 1.3 (page 3), the function f is defined for $x \to a^+$ only if for some μ the interval $(a, a + \mu)$ is a subset of the domain of f.*
Similarly, the function f is defined for $x \to a^-$ only if for some μ the interval $(a - \mu, a)$ is a subset of the domain of f.
Hence, the function f is defined for $x \to a$ only if for some μ the set $(a - \mu, a) \cup (a, a + \mu)$ is a subset of the domain of f.

The **Negation of Definition 1.6** follows from the negation of Definition 1.4 (page 8).

a. A function $f(x)$ is undefined at the right of a, if for every positive number $\mu_+ > 0$

$$f(x) \qquad \text{is not defined for some} \quad 0 < x - a < \mu_+.$$

b. A function $f(x)$ is undefined at the left of a, if for every positive number $\mu_- > 0$

$$f(x) \qquad \text{is not defined for some} \quad 0 < a - x < \mu_-.$$

c. A function $f(x)$ is undefined at either right or left of a, if for every positive number $\mu > 0$

$$f(x) \qquad \text{is not defined for some} \quad 0 < |x - a| < \mu.$$

We apply these negations in the next two examples.

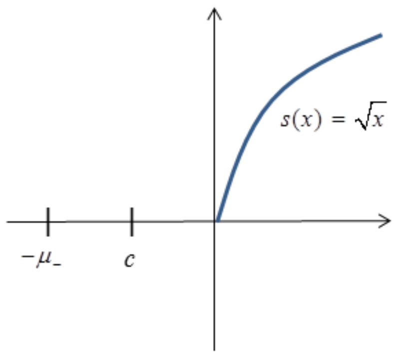


Figure 1.6: The function $s(x) = \sqrt{x}$ is undefined at c

Example 1.7. Figure 1.6 shows the graph of the function $s(x) = \sqrt{x}$.

For every positive number $\mu_- > 0$ there is a number $-\mu_- < c = -\frac{\mu}{2} < 0$, so that

$$s(c) = \sqrt{c} \quad \text{is undefined.}$$

By part (b) of the negation of Definition 1.6, the function

$$s(x) = \sqrt{x} \quad \text{is undefined for } x \to 0^-.$$

$\square$

Example 1.8. The domain of the function

$$f(x) = \cos^{-1} x \quad \text{is the interval } [-1, 1].$$

Hence, by Remark 1.7, it is defined neither at the left of -1 nor at the right of 1. $\square$

Finite Limits at the Right of a Number

Intuitively, the number R is the limit of the function $f(x)$ at the number a from the right, if

> we can make the values of $f(x)$ to be very close to R (as close as we like) by taking numbers x sufficiently close to a from the right.

We must be made clear that this is *not* the definition of the limit

$$\lim_{x \to a^+} f(x) = R.$$

It is neither an "intuitive definition." It merely expresses how we interpret the limit of a function f at a number a from the right. Our task is to express this idea in precise mathematical terms. That is, we should *define* it.

Remark 1.8. *Take note that when we say that we can take x sufficiently close to a from the right, we are assuming that such x exists. See number 3 of Remark 1.3 (page 3)*

Let us consider the following three cases.

In Figure 1.7, $f(x) > R$ for all numbers x on the right of a; thus, the distance between $f(x)$ and R is

$$f(x) - R > 0 \quad \text{for every } x > a.$$

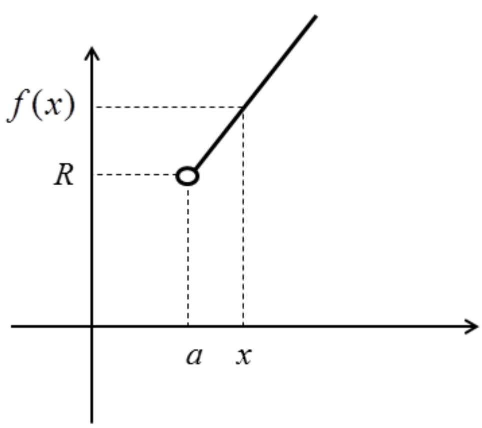


Figure 1.7: $f(x) > R$ for all x on the right of a

In Figure 1.8, $f(x) < R$ for all numbers x on the right of a; thus, the distance between $f(x)$ and R is

$$R - f(x) > 0 \quad \text{for every } x > a.$$

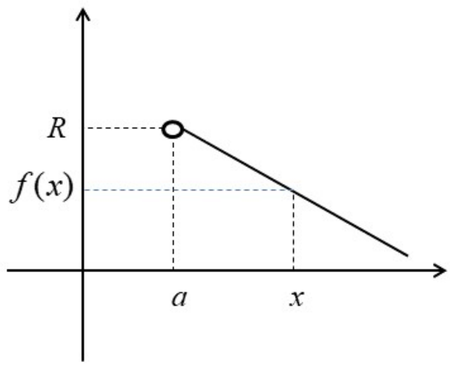


Figure 1.8: $f(x) < R$ for all x on the right of a

Finally, in Figure 1.9, $f(x) < R$ and $f(x) > R$ for all numbers x on the right of a. Hence, the distance between $f(x)$ and R is

$$|f(x) - R| > 0 \quad \text{for every } x > a,$$

because the distance may be either $f(x) - R > 0$ or $R - f(x) > 0$.

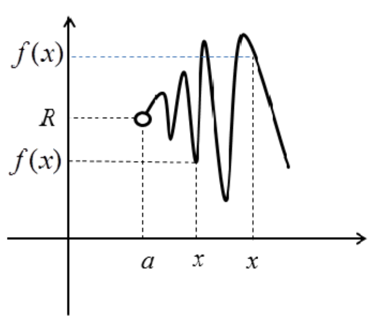


Figure 1.9: $f(x) > R$ and $f(x) < R$ for some x on the right of a

These three particular cases show that the behaviour of a function varies around the number R. That is, the values $f(x)$ might be on the left or right of R.
Hence, in general, the distance between $f(x)$ and R is $|f(x) - R|$.

To express mathematically our understanding of a limit, we start with the statement

> "we can make the values of $f(x)$ to be very close to R (as close as we like)."

The values $f(x)$ are very close to R if the distance $|f(x) - R|$ between $f(x)$ and R is very small; smaller than a very small positive number $\varepsilon > 0$. Thus,

$$|f(x) - R| < \varepsilon.$$

In the figure below we see that the smaller the number ε, the closer is $f(x)$ to R.

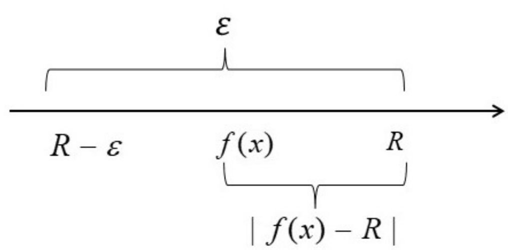


Figure 1.10: $|f(x) - R| < \varepsilon$

If we want the distance between $f(x)$ and R to be small

> "as small as we (you, I and anyone else) like,"

then the number ε must be arbitrarily small, in other words, we want the number ε to be *any* positive number.

The distance between $f(x)$ and R is made very close by taking

> "numbers x sufficiently close to a from the right."

We said earlier that a number x is close to a number a from the right if there is a positive number $\delta^+ > 0$ such that $x - a < \delta^+$.

These two ideas take us to the next definition.

Definition 1.9. The *limit of a function* $f(x)$ *at a number* a *from the right is* R, if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - R| < \varepsilon \quad \text{for every} \quad 0 < x - a < \delta. \tag{1.1}$$

The limit is represented as either

$$\lim_{x \to a^+} f(x) = R \qquad \text{or} \qquad f(x) \to R \quad \text{as} \quad x \to a^+. \tag{1.2}$$

Figure A.1 (page 262) shows the graphic representation of Definition 1.9.

We have from the statement (1.1) that if

$$0 < x - a < \delta \quad \text{a number } x \text{ is in the interval } (a, a+\delta),$$

then

$$|f(x) - R| < \varepsilon \quad \text{the distance from } f(x) \text{ to } R \text{ is less than } \varepsilon.$$

Hence, the statement (1.1) only makes sense if

- the function $f(x)$ is defined for $x \to a^+$ (see part (a) of Definition 1.6 on page 10), and
- the number R exists.

By the former we should only consider limits of functions which are defined at a from the right. Otherwise, the limit is undefined. That is,

$$\lim_{x \to a^+} f(x) \quad \text{is undefined if } f(x) \text{ is undefined for } x \to a^+.$$

By the latter the limit (1.2) exists only if it is equal to a number R. Hence,

$$\lim_{x \to a^+} f(x) \qquad \text{does not exist if} \qquad \lim_{x \to a^+} f(x) \neq R \quad \text{for every number } R.$$

This take us to the negation of Definition 1.9. Intuitively, the limit (1.2) does not exist

> if for any number R, there is a number x close to the number a from the right, such that $f(x)$ is not close to R.

The negation of the statement (1.1) formalizes this idea.

Negation of Definition 1.9

The limit $\lim_{x \to a^+} f(x)$ does not exist
if for every number R, there is a positive number $\varepsilon > 0$, such that for every positive number $\delta > 0$

$$|f(x) - R| \geq \varepsilon \qquad \text{for some} \qquad 0 < x - a < \delta. \tag{1.3}$$

The function of the next example will be used to provide examples and counterexamples, throughout this *guide*.

Example 1.9. We can see from the graph of the function

$$\sin\left(\frac{1}{x}\right) \qquad \text{shown in Figure 1.11,}$$

that the side limits at zero of this function do not exist, but seeing is not knowing, we must apply the statement (1.3) to prove it.

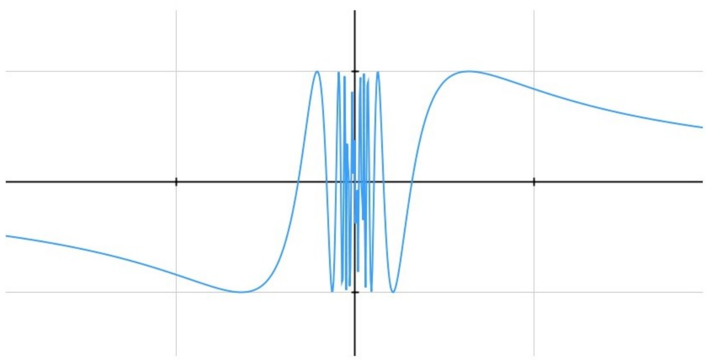


Figure 1.11: Graph of the function $\sin\left(\frac{1}{x}\right)$

We must show that for any number R, there is a positive number $\varepsilon > 0$ such that for every positive number $\delta > 0$

$$\left|\sin\left(\frac{1}{x}\right) - R\right| \geq \varepsilon \quad \text{for some} \quad 0 < x < \delta. \tag{1.4}$$

We use the facts that for any positive number $\delta > 0$, there are positive numbers $M, N > 0$ such that

$$\left|\frac{2}{(4x+1)\pi}\right| < \delta \qquad \text{for every} \qquad x > N, \tag{1.5}$$

and

$$\left|\frac{6}{(12x+1)\pi}\right| < \delta \qquad \text{for every} \qquad x > M. \tag{1.6}$$

These statements will be made evident in Example 5.10 (page 111).

If $k > N$ is an integer, then by (1.5) the number

$$x = \frac{2}{(4k+1)\pi}$$

is such that

$$0 < x < \delta \quad \text{and} \quad 0 < \frac{1}{x} = \frac{(4k+1)\pi}{2}.$$

For this number x we have

$$\sin\left(\frac{1}{x}\right) = \sin\left(\frac{(4k+1)\pi}{2}\right) = 1. \tag{1.7}$$

By inequalities 1 and 3 on page xii, for any number R

$$\left|\sin\left(\frac{1}{x}\right) - R\right| \geq \left|\sin\left(\frac{1}{x}\right)\right| - |R| \tag{1.8}$$

$$\left|\sin\left(\frac{1}{x}\right) - R\right| \geq |R| - \left|\sin\left(\frac{1}{x}\right)\right| \tag{1.9}$$

We consider three cases.

Case 1. If $|R| > 1$, we have the positive number $\quad \varepsilon = |R| - 1 > 0$.

By the statement (1.7) and inequality (1.9), we have that for this positive number ε and every positive number $\delta > 0$ there is

$$0 < x = \frac{2}{(4k+1)\pi} < \delta,$$

so that

$$\left|\sin\left(\frac{1}{x}\right) - R\right| \geq |R| - \left|\sin\left(\frac{1}{x}\right)\right| = |R| - 1 = \varepsilon.$$

Case 2. If $|R| < 1$, we have the positive number $\varepsilon = 1 - |R| > 0$.

By the statement (1.7) and inequality (1.8), we have that for this positive number ε and every positive number $\delta > 0$ there is

$$0 < x = \frac{2}{(4k+1)\pi} < \delta,$$

so that

$$\left|\sin\left(\frac{1}{x}\right)-R\right| \geq \left|\sin\left(\frac{1}{x}\right)\right|-|R| = 1-|R| = \varepsilon.$$

Case 3. If $|R| = 1$, we have the positive number $\varepsilon = \frac{1}{2}$.

Similarly, if $k > M$ is an integer, then by (1.6) the number

$$y = \frac{6}{(12k+1)\pi}$$

is such that

$$0 < y < \delta \quad \text{and} \quad 0 < \frac{1}{y} = \frac{(12k+1)\pi}{6}.$$

For this number y we have

$$\sin\left(\frac{1}{y}\right) = \sin\left(\frac{(12k+1)\pi}{6}\right) = \frac{1}{2}. \tag{1.10}$$

By the statement (1.10) and inequality (1.9), we have that for this positive number ε and every positive number $\delta > 0$ there is

$$0 < y = \frac{3}{(6k+1)\pi},$$

so that

$$\left|\sin\left(\frac{1}{y}\right)-R\right| \geq |R|-\left|\sin\left(\frac{1}{y}\right)\right| = 1-\frac{1}{2} = \frac{1}{2} = \varepsilon.$$

In either case, the statement (1.4) holds, and the limit from the right is not equal to any real number R. $\square$

From this example, we see that it may not be trivial to apply statement (1.3) to prove that a side limit does not exist.

Finite Limits at the Left of a Number

Similarly, the number L is the limit of the function $f(x)$ at the number a from the left, if

> we can make the values of $f(x)$ to be very close to L (as close as we like) by taking numbers x sufficiently close to a from the left.

Again, this is *not* the definition of the limit

$$\lim_{x \to a^-} f(x) = L.$$

It merely expresses how we interpret the limit of a function f at a number a from the left.

As in the previous case, the behaviour of a function varies around the number L. The values $f(x)$ might be on the left, right or both sides of L. Hence, again the distance between $f(x)$ and L is $|f(x) - L| > 0$.

The definition of a limit at a number from the left is similar to Definition 1.9 (page 15).

Definition 1.10. The *limit of a function* $f(x)$ *at a number* a *from the left is* L, if for any positive number $\varepsilon > 0$ there is a positive number $\delta^- > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for every} \quad 0 < a - x < \delta^- \tag{1.11}$$

and it is represented as either

$$\lim_{x \to a^-} f(x) = L \qquad \text{or} \qquad f(x) \to L \quad \text{as} \quad x \to a^-. \tag{1.12}$$

Figure A.2 (page 263) shows the graphic representation of Definition 1.10.

We have from the statement (1.11) that if

$$0 < a - x < \delta^- \quad \text{a number } x \text{ is in the interval } (a - \delta, a),$$

then

$$|f(x) - L| < \varepsilon \quad \text{the distance from } f(x) \text{ to } L \text{ is less than } \varepsilon.$$

Hence, the statement (1.11) only makes sense if

- the function $f(x)$ is defined for $x \to a^-$ (see part (b) of Definition 1.6 (page 10)), and
- the number L exists.

By the former we should only consider limits of functions which are defined at a from the left. Otherwise, the limit is undefined. That is,

$$\lim_{x \to a^-} f(x) \quad \text{is undefined if } f(x) \text{ is undefined for } x \to a^-.$$

By the latter the limit (1.12) exists only if it is equal to a number L. Hence,

$$\lim_{x \to a^-} f(x) \qquad \text{does not exist if} \qquad \lim_{x \to a^-} f(x) \neq L \quad \text{for every number } L.$$

This take us to the negation of Definition 1.10. Intuitively, the limit (1.12) does not exist

> if for every number L, there is a number x close to the number a from the right such that $f(x)$ is not close to L.

The negation of the statement (1.11) expresses this idea.

Negation of Definition 1.10

The limit $\lim\limits_{x \to a^-} f(x)$ does not exist
if for every number L, there is a positive number $\varepsilon > 0$, such that for every positive number $\delta > 0$

$$|f(x) - L| \geq \varepsilon \qquad \text{for some} \qquad 0 < a - x < \delta. \tag{1.13}$$

If students are not taught how to apply statements (1.3) and (1.13) in order to prove that a side limit does not exist, then the least they should know is that a side limit does not exist if it is not equal to a real number.

It is not true that side limits always exist as Example 1.9 and Exercise 5 on page 26 show.

Finite Limits at a Number

Intuitively, the number M is the limit of the function $f(x)$ at the number a, if

> we can make the values of $f(x)$ to be very close to M (as close as we like) by taking numbers x sufficiently close to a.

As in the case of side limits, this is *not* the definition of the limit

$$\lim_{x \to a} f(x) = M. \tag{1.14}$$

It merely expresses how we interpret the limit of a function f at a number a.

The definition of the limit (1.14) is as follows.

Definition 1.11. The *limit of a function $f(x)$ at a number a is M*, if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - M| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta. \tag{1.15}$$

It is represented as either

$$\lim_{x \to a} f(x) = M \qquad \text{or} \qquad f(x) \to M \quad \text{as} \quad x \to a. \tag{1.16}$$

Figure A.3 (page 263) shows the graphic representation of Definition 1.11.

We have from the statement (1.15) that if

$0 < |x - a| < \delta$ a number x is in the union of intervals $(a - \delta, a) \cup (a, a + \delta)$,

then

$|f(x) - M| < \varepsilon$ the distance from $f(x)$ to M is less than ε.

Compare Definition 1.11 with Definition 1.9 (page 15), and Definition 1.10 (page 20).

Again, the statement (1.15) only makes sense if

- the function $f(x)$ is defined for $x \to a$ (see part (c) of Definition 1.6 on page 10), and
- the number M exists.

By the former we should only consider limits of functions which are defined around a. Otherwise, the limit is undefined. That is,

$$\lim_{x\to a} f(x) \quad \text{is undefined if } f(x) \text{ is undefined for } x \to a.$$

By the latter the limit (1.16) exists only if it is equal to a number M. Hence,

$$\lim_{x\to a} f(x) \qquad \text{does not exist if} \qquad \lim_{x\to a^+} f(x) \neq M \quad \text{for every number } M.$$

This take us to the negation of Definition 1.11.

Intuitively, the limit (1.16) does not exist

> if for every number M, there is a number x close to a number a such that $f(x)$ is not close to M.

The negation of the statement (1.15) formalizes this idea.

Negation of Definition 1.11.

The limit $\lim_{x\to a} f(x)$ does not exist
if for every number M there is a positive number $\varepsilon > 0$, such that for every positive number $\delta > 0$

$$|f(x) - M| \geq \varepsilon \qquad \text{for some} \qquad 0 < |x - a| < \delta. \tag{1.17}$$

The relationship between the side limits (1.2) and (1.12), and limit at a number (1.16) is established in the next preposition.

Proposition 1.12. *The limit* $\lim_{x\to a} f(x) = M$ *exists* *if and only if*

a. $\lim_{x\to a^+} f(x) = R$ *exists,*

b. $\lim_{x\to a^-} f(x) = L$ *exists, and*

c. $R = L = M$.

Proof. **(⇒)** For any positive number $\varepsilon > 0$, there is a positive number $\delta > 0$ such that

$$|f(x) - M| < \varepsilon \quad \text{for all} \quad 0 < |x - a| < \delta \Leftrightarrow -\delta < x - a < \delta$$

Hence,

$$|f(x) - M| < \varepsilon \quad \text{for every} \quad 0 < x - a < \delta \text{ and } 0 < a - x < \delta.$$

Therefore

$$\lim_{x\to a^+} f(x) = M = \lim_{x\to a^-} f(x).$$

(⇐) If

$$\lim_{x\to a^+} f(x) = R \quad \text{and} \quad \lim_{x\to a^+} f(x) = L,$$

for any positive number $\varepsilon > 0$, there are positive numbers $\delta_1 > 0$ and $\delta_2 > 0$, such that

$$|f(x) - R| < \frac{\varepsilon}{2} \quad \text{for} \quad 0 < x - a < \delta_1$$

and

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{for} \quad 0 < a - x < \delta_2.$$

For the positive number $\delta = \min(\delta_1, \delta_2) > 0$, we have

$$x - a < \delta \leq \delta_1 \quad \text{and} \quad a - x < \delta \leq \delta_2.$$

Thus,

$$|f(x) - R| < \frac{\varepsilon}{2} \quad \text{and} \quad |f(x) - L| < \frac{\varepsilon}{2} \quad \text{for every } 0 < |x - a| < \delta.$$

We show next, that $R = L$. By the triangle inequality on page xii

$$|R - L| = |f(x) - L + R - f(x)| \leq |f(x) - R| + |f(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since this is for any positive number $\varepsilon > 0$ we conclude that $|R - L| = 0$ (see Exercise 15 of Chapter 3).

Moreover,

$$|f(x) - R| < \frac{\varepsilon}{2} < \varepsilon \quad \text{for } 0 < |x - a| < \delta.$$

Therefore, the limit $\lim_{x\to a} f(x)$ exists and it is equal to $R = L$.

Q.E.D.

Therefore, a limit at a number does not exist if and only if the conclusion of Proposition 1.12 does not hold.

Negation of Proposition 1.12 (page 24).

The limit $\lim_{x\to a} f(x)$ does not exist if and only if either

a. $\lim_{x\to a^+} f(x)$ does not exist, or

b. $\lim_{x\to a^-} f(x)$ does not exist, or

c. these two side limits exist but they are not equal.

By Example 1.9 (page 17) the limit

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist}$$

because part (a) of the above conclusion does not hold.

There are two options to prove that a limit does not exist, either we prove the statement (1.17) (page 23), or any one of the parts (a), (b) or (c) of the negation of Proposition 1.12.

Most students find the former difficult if not impossible to do. However, part (c) of the latter is within their capabilities. In general, this is what they are expected to do when they are asked to explain why a limit does not exist. In such instances, students should be asked to do precisely this. That is, they must be asked to show that a limit does not exist because the side limits exist but they are not equal, as in the next example.

Example 1.10. We consider the piecewise function

$$f(x) = \begin{cases} x & \text{if} \quad x > 1 \\ 0 & \text{if} \quad -1 < x < 1. \end{cases}$$

The side limits exist at the number 1,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x = 1 \qquad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0 = 0.$$

Parts (a) and (b) of the negation above do not hold but part (c) does. Hence,

$$\lim_{x \to 1} f(x) \qquad \text{does not exists.}$$

$\square$

Exercises I

1. Explain why the set

$$x \to a^+ = \left\{ x \in \mathbb{R} \mid 0 < x - a < \frac{1}{2} \right\}$$

has infinitely many elements.

2. Express, mathematically, the statements listed below.

   a. There is a number very closed to 2.
   b. The distance between the numbers a and b is less that 10^{-6}.
   c. We can always find a rational number very close to π.

3. Prove that the function

$$\sin\left(\frac{1}{x}\right) \qquad \text{is neither positive nor negative for } x \to 0^-.$$

4. Prove that the function $\csc\left(\dfrac{1}{x}\right)$ is not defined at 0 from the left.

5. Give an example of a function $f(x)$ and a number a such that the limit

$$\lim_{x \to a^+} f(x) \quad \text{is undefined.}$$

6. Prove that the limit

$$\lim_{x \to 0^-} \sin\left(\frac{1}{x}\right) \quad \text{does not exist.}$$

Hint. Assume that for every positive number $\delta > 0$, there are negative numbers $N, M < 0$, such that

$$\left|\frac{2}{(4x-1)\pi}\right| < \delta \qquad \text{for every} \qquad x < N. \tag{1.18}$$

and

$$\left|\frac{6}{(12x-1)\pi}\right| < \delta \qquad \text{for every} \qquad x < M. \tag{1.19}$$

See Exercise 4 and Exercise 5 of chapter 5.

7. Prove that $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a}(f(x) - L) = 0$.

8. Prove that if $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} |f(x)| = |L|$.

9. Give a counterexample to show that the converse of Exercise 7 is not true. That is, give a function $f(x)$ and a number a such that

$$\lim_{x \to a} |f(x)| = |L| \quad \text{and} \quad \lim_{x \to a} f(x) \neq L.$$

10. Let $f(x)$ be the function

$$f(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad 0 < x < \pi \end{cases}$$

Explain why the limit

$$\lim_{x \to 0} f(x) \quad \text{does not exist}$$

and

$$\lim_{x \to \pi} f(x) \quad \text{is undefined}$$

11. a. Use the definition of the sine function to prove that

$$|\sin\alpha - \sin\theta| \leq |\alpha - \theta| \quad \text{for every } 0 \leq \alpha, \theta \leq \frac{\pi}{2}.$$

b. Use part (a) to prove that

$$\lim_{x\to a} \sin x = \sin a \quad \text{for every } 0 \leq a \leq \frac{\pi}{2}.$$

Complete solutions are provided on page 275.

Chapter 2
Finite Limits at Infinity

Misunderstanding arise whenever we leave the evaluation of limits at infinity to our intuition. We must formalize what we mean by numbers which increase and decrease "infinitely."

The infinity symbol "∞" requires its correct understanding of what it means and what it does not. We use this symbol to convey the intuitive idea of large (positive) numbers and small (negative) numbers.

The infinity symbol does *not* represent a quantity, so it can never, never be treated as a real number.
Arithmetic operations involving the infinity symbol do not make sense.

Infinitely Positive Large Numbers

We say that a positive number is infinitely large if its distance to zero is very large. If $V > 0$ is a very large number and the number x is bigger than V, then x is also a very large number.
That is, all numbers on the right of V are "infinitely" large.

For *some* positive large number V (it does not matter how large) there are always numbers larger than V. Hence, the expression $x \to \infty$ represents the set of all numbers larger than V.

$$x \to \infty = \{x \in \mathbb{R} \mid x > V \quad \text{for some} \quad V > 0\}.$$

This is what we mean by "numbers which increase without bounds."
The expression $x \to \infty$ reads "x *tends* to infinity."

Remark 2.1. *1. In interval notation $x \to \infty$ is the open infinite interval (V, ∞).*

2. It is clear that the set $x \to \infty$ is nonempty.

It is incorrect to read $x \to \infty$ as "x approaches infinity."

On the real line, there is no place for infinity, thus a number cannot approach something which is nowhere.

Similarly to Definition 1.4 (page 4), we have the following.

Definition 2.2. A function f has the *property P for infinitely large numbers x* if for some positive number $V > 0$

$$f(x) \quad \text{has the property } P \text{ for every } x > V.$$

We write

$$f(x) \qquad \text{has the property } P \text{ for } x \to \infty. \tag{2.1}$$

In the next example the property P is "positive."

Example 2.1. Figure 2.1 shows the graph of the function $f(x) = x^3$.
We *see* clearly that this function is positive for $x \to \infty$.

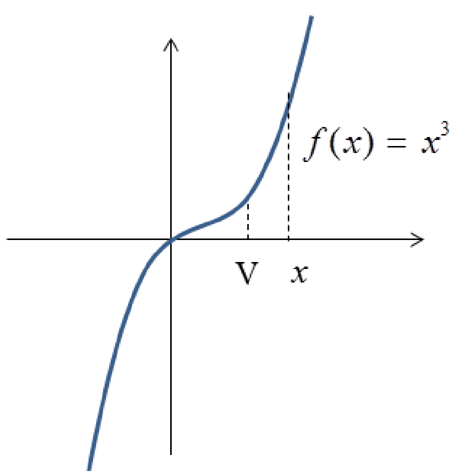


Figure 2.1: $f(x) = x^3 > 0$ for $x > 1$

Indeed, for the positive number $V = 1 > 0$ and every $x > V = 1$ we have, by rule 4 of the Rules of Inequalities (page xii) that

$$x^2 > x > 1 \quad \Rightarrow \quad x^3 > x > 1 > 0.$$

Thus, by Definition 2.2

$$f(x) = x^3 > 0 \quad \text{for } x \to \infty.$$

$\square$

Identify the property P in the next example.

Example 2.2. For the positive number $V = 1 > 0$ the natural logarithm function

$$\ln x \quad \text{is defined for every number } x > V.$$

Hence, $\ln x$ is defined for $x \to \infty$. □

Negation of Definition 2.2.

A function f does not have the property P for $x \to \infty$, if for every positive number $V > 0$

$$f(x) \quad \text{does not have the property } P \text{ for some} \quad x > V.$$

Example 2.3. The tangent function is not defined for $x \to \infty$.
Indeed, for any positive number $V > 0$ there is an integer

$$k > \frac{V}{2\pi} > 0 \quad \Rightarrow \quad x = 2\pi k > V$$

and

$$\tan(x) = \tan(2\pi k) \quad \text{is undefined.}$$

□

Infinitely Negative Small Numbers

Infinitely small numbers are negative and very far from zero. If $U < 0$ is a very small negative number and the number x is less than U, then x is also a very small number. That is, all numbers x on the left of U are "infinitely" small.

For *every* negative small U (it does not matter how small) there are always numbers smaller than U. Hence, the expression $x \to -\infty$ represents the set of all numbers smaller than U.

$$x \to -\infty = \{x \in \mathbb{R} \,|x < U \quad \text{for some} \quad U < 0\}.$$

This is what we mean by "numbers which decrease without bounds." The expression $x \to -\infty$ reads "x *tends* to negative infinity."

The negative sign in the expression $-\infty$ should remaind us of the fact that small numbers are *negative*.

Remark 2.3. *1. In interval notation $x \to -\infty$ is the open infinite interval $(-\infty, U)$.*

2. It is clear that the set $x \to -\infty$ is nonempty.

It is also incorrect to read $x \to -\infty$ as "x approaches negative infinity."

On the real line, there is no place for negative infinity either, thus a number cannot approach something which is nowhere.

Compare the following definition with Definition 2.2 (page 30).

Definition 2.4. A function f has the *property P for infinitely small numbers x* if for some negative number $U < 0$

$$f(x) \quad \text{has the property } P \text{ for all } x < U.$$

We write

$$f(x) \qquad \text{has the property } P \text{ for } x \to -\infty. \tag{2.2}$$

The next example is similar to Example 2.1 (page 30).

Example 2.4. Consider the function $f(x) = x^3$ whose graph is shown in Figure 2.1 (page 30).
For the negative number $U = -1$ and every number $x < U = -1$, we have by rule 5 of the Rules of Inequalities (page xii)

$$x < -1 \quad \Rightarrow \quad -x > 1 \quad \text{and} \quad x^2 > -x > 1 \quad \Rightarrow \quad x^3 < x < -1.$$

Thus, $f(x) = x^3 < 0$ for every $x < U$, and the function

$$f(x) = x^3 \quad \text{is negative for } x \to -\infty.$$

$\square$

Negation of Definition 2.4

A function f does not have the property P for $x \to -\infty$, if for every negative number $U < 0$

$$f(x) \quad \text{does not have the property } P \text{ for some} \quad x < U.$$

In the next example, we prove two well-known facts about the sine function.

Example 2.5. For every positive number $V > 0$ and every negative number $U < 0$, there are two integers k and n so that

$$k > \frac{V}{2\pi} > 0 \quad \text{and} \quad n < \frac{U}{2\pi} < 0.$$

Thus, for

$$u = 2k\pi > V \quad \text{and} \quad v = 2n\pi < U$$

we have that

$$\sin u = 0 \quad \text{and} \quad \sin v = 0.$$

By the negations of Definition 2.2 and Definition 2.4, for $x \to \infty$ and for $x \to -\infty$ the function $\sin x$ is neither positive nor negative. □

Finite Limits at Infinity: Horizontal Asymptotes

Intuitively, the number K is the limit of the function $f(x)$ for infinitely large numbers x if

> we can make the values of $f(x)$ to be very close to K (as close as we like) by taking infinitely large numbers x.

This statement is *not* the definition of the limit

$$\lim_{x\to\infty} f(x) = K. \tag{2.3}$$

It merely expresses how we interpret the limit (2.3).

Let us consider the three cases shown in Figures 2.2 to 2.4.

In Figure 2.2, for some positive number $V > 0$,

$$f(x) < K \quad \text{for every } x > V.$$

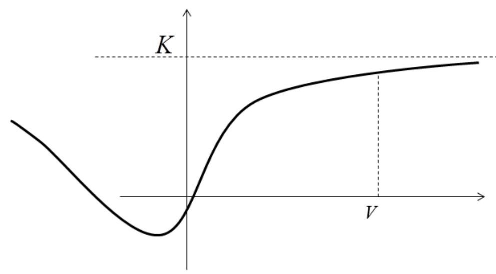


Figure 2.2: $f(x) < K$ for $x \to \infty$

In Figure 2.3, for some positive number $V > 0$,

$$f(x) > K \quad \text{for every } x > V.$$

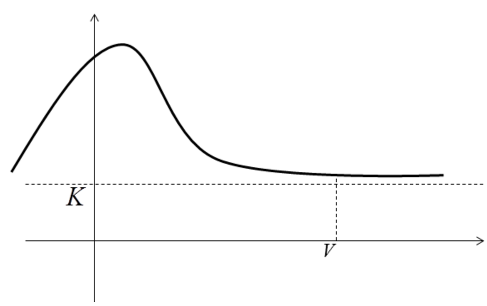


Figure 2.3: $f(x) > K$ for $x \to \infty$

The graphs in Figure 2.2 and Figure 2.3 do not cross the horizontal lines $y = K$ for $x \to \infty$.

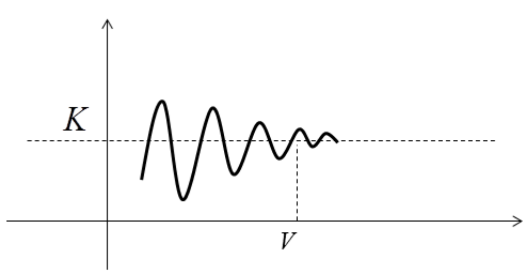


Figure 2.4: $f(x) > K$ and $f(x) < K$ for $x \to \infty$

For some positive number $V > 0$,

$$f(x) > K \quad \text{and} \quad f(x) < K \quad \text{for every } x > V.$$

The graph does cross the horizontal line $y = K$ for $x \to \infty$.

In Figures 2.2 to 2.4 (pages 34 to 34) the horizontal line $y = K$ is a horizontal asymptote of the function f in the positive direction.

From Figure 2.4 we see that it is incorrect to characterize a horizontal asymptote as "the line the function f gets close to but never touches it."

In the previous chapter we established the idea of

"$f(x)$ is very close to K (as close as we like)"

as: for any positive number $\varepsilon > 0$ the distance $|f(x) - K|$ between $f(x)$ and K is less than ε.
The idea of

"taking infinitely large numbers x"

is expressed as: all numbers $x > V$ for some positive number $V > 0$.

Hence, we have the following definition.

Definition 2.5. The *limit at (positive) infinity of a function* $f(x)$ is K if for any positive number $\varepsilon > 0$ there is a positive number $V > 0$ such that

$$|f(x) - K| < \varepsilon \qquad \text{for every} \quad x > V \tag{2.4}$$

and it is represented as either

$$\lim_{x\to\infty} f(x) = K \qquad \text{or} \qquad f(x) \to K \quad \text{as} \quad x \to \infty. \tag{2.5}$$

Figure A.4 (page 264) shows the graphic representation of Definition 2.5.

We have from the statement (2.4) that if

$x > V$ a number x is greater than V,

then

$|f(x) - K| < \varepsilon$ the distance from $f(x)$ to K is less than ε.

Hence, the statement (2.4) only makes sense if

- the function f is defined for $x \to \infty$, and
- the number K exist.

By the former we should only consider limits of functions which are defined on an infinite interval (V, ∞) for some $V > 0$. Otherwise, the limit is undefined. That is,

$$\lim_{x\to\infty} f(x) \quad \text{is undefined if } f(x) \text{ is undefined for } x \to \infty.$$

By the latter, the limit (2.5) only exists if it is equal to a number K. Hence,

$$\lim_{x\to\infty} f(x) \quad \text{does not exist if} \quad \lim_{x\to\infty} f(x) \neq K \quad \text{for every number } K. \tag{2.6}$$

Example 2.6. The natural logarithm function is defined for all $x > 0$; hence, the limit

$$\lim_{x\to\infty} \ln x \quad \text{is well defined.}$$

In contrast, by Example 2.3 (page 31), the tangent function is not defined for every infinite interval. Hence, the limit

$$\lim_{x\to\infty} \tan x \quad \text{is undefined.}$$

□

In the next example we provide a discontinuous function whose limit at infinity exists.

In other words, the existence of the limit of a function at infinity does not imply the continuity of the function.

Example 2.7. Consider the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x \in \mathbb{N} \\ 0 & \text{if} \quad x \notin \mathbb{N} \end{cases}$$

defined on $(0, \infty)$. Its graph is shown in Figure 2.5.

Visually, the limit of this function at infinity is zero. Indeed, for every positive number $\varepsilon > 0$, we have the positive number $V = \dfrac{1}{\varepsilon} > 0$. If

$$x > V \quad \Rightarrow \quad \frac{1}{x} < \frac{1}{V} = \varepsilon.$$

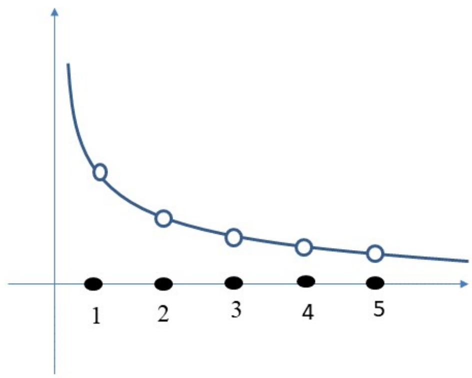


Figure 2.5: Graph of the discontinuous function $f(x)$

Thus, $f(x) = 0$ or $f(x) = \dfrac{1}{x}$ and in either case $|f(x)| < \varepsilon$. By Definition 2.5 (page 35)

$$\lim_{x\to\infty} f(x) = 0.$$

$\square$

Example 2.8. Similarly to the previous example we have that for every positive number $\varepsilon > 0$, we have the positive number $V = \dfrac{1}{\varepsilon} > 0$ such that

$$\left|\frac{1}{x}\right| < \varepsilon \quad \text{for every } x > V = \frac{1}{\varepsilon}.$$

Hence,

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

$\square$

Negation of Definition 2.5 (page 35)

The limit

$$\lim_{x\to\infty} f(x) \neq K \quad \text{does not exist,}$$

if for any number K, there is a positive number $\varepsilon > 0$ such that for every positive number $V > 0$

$$|f(x) - K| \geq \varepsilon \quad \text{for some} \quad x > V. \tag{2.7}$$

From the graphs of the sine and cosine functions, we easily see that

$$\lim_{x\to\infty} \sin x \quad \text{and} \quad \lim_{x\to\infty} \cos x \quad \text{do not exist.}$$

But their proofs are not trivial, as we show in the next example.

Example 2.9. To show that the limit

$$\lim_{x\to\infty} \sin x \neq K \quad \text{for every } K,$$

we take any positive number $V > 0$ and show that there is a positive number $\varepsilon > 0$ such that (2.7) holds.

For any number K, we consider two cases.

Case 1. If $K \neq 0$, we have the positive number $\varepsilon = |K| > 0$. Thus, for every positive number $V > 0$ there is an integer

$$k > \frac{V}{2\pi} \quad \Rightarrow \quad x = 2k\pi > V$$

so that

$$|\sin x - K| = |K| = \varepsilon.$$

Case 2. If $|K| = 0$, we have the positive number $\varepsilon = 1$. Thus, for every positive number $V > 0$, there is an integer

$$k > \frac{2V-1}{2\pi} \quad \Rightarrow \quad x = \frac{(2k+1)\pi}{2} > V$$

so that

$$|\sin x - K| = 1 = \varepsilon.$$

In either case the statement (2.7) holds and the limit

$$\lim_{x\to\infty} \sin x \quad \text{does not exist.}$$

□

Similarly, we have that a number K is the limit of a function $f(x)$ for infinitely small numbers if we can make the values of $f(x)$ to be very close to K (as close as we like) by taking infinitely small numbers x.

This statement is *not* the definition of the limit

$$\lim_{x\to-\infty} f(x) = K. \tag{2.8}$$

It is an interpretation of the limit (2.8).

Similarly to Figures 2.2 to 2.4 (pages 34 to 34) we have three cases.

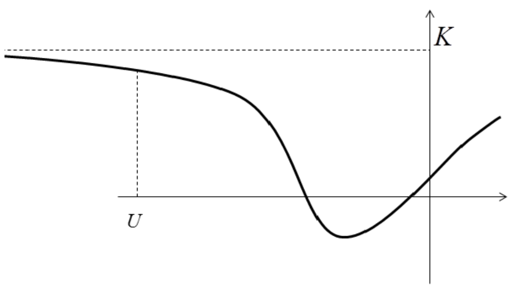


Figure 2.6: $f(x) < K$ for $x \to -\infty$

In Figure 2.7, for some negative number $U < 0$

$$f(x) < K \quad \text{for all numbers } x < U\ .$$

The graph of the function f does not crosses the horizontal line $y = K$ for $x \to \infty$.

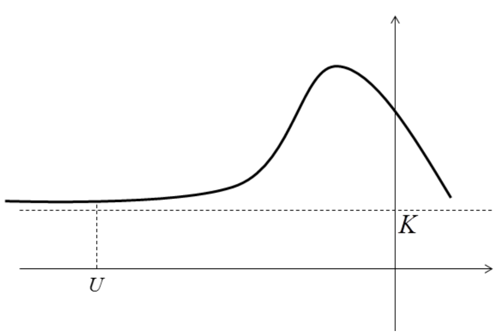


Figure 2.7: $f(x) > K$ for $x \to -\infty$

In Figure 2.8, for some negative number $U < 0$

$$f(x) > K \quad \text{for all numbers } x < U.$$

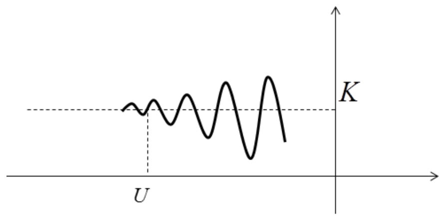


Figure 2.8: $f(x) > K$ and $f(x) < K$ for $x \to -\infty$

The graph of the function f does not crosses the horizontal line $y = K$ for $x \to \infty$.

In Figure 2.8 for some negative number $U < 0$

$$f(x) > K \quad \text{and} \quad f(x) < K \quad \text{for every } x < U.$$

It is clear that the function f does cross the horizontal line $y = K$ for $x \to \infty$.

In Figures 2.6 to 2.8 the horizontal line $y = K$ is a horizontal asymptote of the function f in the negative direction.

Since the idea of "taking infinitely small numbers x" is expressed as: $x < U$ for some negative number $U < 0$, we have the following definition.

Definition 2.6. The *limit at (negative) infinity of a function* $f(x)$ is K if for any positive number $\varepsilon > 0$ there is a negative number $U < 0$ such that

$$|f(x) - K| < \varepsilon \qquad \text{for every} \quad x < U \tag{2.9}$$

and it is represented as either

$$\lim_{x \to -\infty} f(x) = K \qquad \text{or} \qquad f(x) \to K \quad \text{as} \quad x \to -\infty. \tag{2.10}$$

Figure A.5 (page 264) shows the graphic representation of Definition 2.6.

We have from the statement (2.10) that if

$$x < U \quad \text{a number } x \text{ is smaller than } V,$$

then

$$|f(x) - K| < \varepsilon \quad \text{the distance from } f(x) \text{ to } K \text{ is less than } \varepsilon.$$

Hence, the statement (2.9) only makes sense if

- the function f is defined for $x \to -\infty$, and
- the number K exist.

By the former we should only consider limits of functions which are defined on an infinite interval $(-\infty, U)$ for some $U < 0$. Otherwise, the limit is undefined. That is,

$$\lim_{x \to -\infty} f(x) \quad \text{is undefined if } f(x) \text{ is undefined for } x \to -\infty.$$

By the latter, the limit (2.10) only exists if it is equal to a number K. Hence,

$$\lim_{x \to -\infty} f(x) \quad \text{does not exist if} \quad \lim_{x \to -\infty} f(x) \neq K \quad \text{for every number } K. \tag{2.11}$$

Example 2.10. The natural logarithm function $\ln|x|$ is defined for all $x < 0$, hence the limit

$$\lim_{x \to -\infty} \ln|x| \quad \text{is well defined.}$$

In contrast, the tangent function is not defined for every infinite interval. That is, for every negative number $U < 0$ there is a number x [which one?] such that $\tan x$ is undefined. Hence, the limit

$$\lim_{x \to -\infty} \tan x \quad \text{is undefined.}$$

□

In the next examples we prove the existence of limits at negative infinity.

Example 2.11. To show that zero is the limit of

$$\lim_{x \to -\infty} e^x$$

we consider any positive number $\varepsilon > 0$.

Case 1. If $0 < \varepsilon < 1$, we have the negative number $N = \ln \varepsilon < 0$. Hence,

$$e^x < e^N = \varepsilon \quad \text{for every } x < N.$$

Case 2. If $\varepsilon \geq 1$, we have the negative number $N = -\ln \varepsilon < 0$. Hence,

$$e^x < e^{-\ln \varepsilon} = \frac{1}{\varepsilon} \leq 1 \leq \varepsilon \quad \text{for every } x < N.$$

By (2.9) above, the limit is 0. □

Example 2.12. For any positive number $\varepsilon > 0$, we have the negative number $U = -\frac{1}{\varepsilon} < 0$ such that

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} = -\frac{1}{x} < \varepsilon \quad \text{for every } x < U = -\frac{1}{\varepsilon}.$$

Hence,

$$\lim_{x \to -\infty} \frac{1}{x} = 0.$$

□

The negation of Definition 2.6 is similar to the negation of Definition 2.5.

Negation of Definition 2.6 (page 40)

The limit

$$\lim_{x \to -\infty} f(x) \neq K \quad \text{does not exist,}$$

if for any number K, there is a positive number $\varepsilon > 0$ such that for every negative number $U < 0$

$$|f(x) - K| \geq \varepsilon \quad \text{for some} \quad x < U. \tag{2.12}$$

The limits

$$\lim_{x \to -\infty} \sin x \quad \text{and} \quad \lim_{x \to -\infty} \cos x \quad \text{do not exist.}$$

Again, from their graphs we <u>see</u> that this is true. See Example 2.9 (page 38).

When students are not capable of applying the negation of Definition 2.6, then they offer arguments based on functions' graphs in order of justify the non-existence of limits at infinity. In these instances, instructors should be made them aware that these type of arguments are not mathematically sound and why.

Students must apply Definition 2.7 to justify the existence of asymptotes. Any other type of arguments should not be acceptable.

Definition 2.7. The horizontal line $y = K$ is a horizontal asymptote of the function f if and only if either

$$\lim_{x\to\infty} f(x) = K \quad \text{or} \quad \lim_{x\to-\infty} f(x) = K.$$

Students must be able to explain the difference of these two limits, in terms of the function's graph.

Exercises II

1. Prove that the sine function is neither positive nor negative for $x \to -\infty$.
2. Prove that the limit
$$\lim_{x\to-\infty} \cos x \quad \text{does not exist.}$$
3. Prove that the limit
$$\lim_{x\to\infty} \delta_C(x) \quad \text{does not exist,}$$
where δ_C is the function defined as
$$\delta_C(x) = \begin{cases} x & if \quad x \in \mathbb{Z} \\ 0 & if \quad x \notin \mathbb{Z} \end{cases}$$
4. Apply Definition 2.5 (page 35) and Definition 2.6 (page 40) to prove that
$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0.$$

Complete solutions are provided on page 284.

Chapter 3
Properties of Finite Limits

From now on, we use the expression $\lim f(x)$ to indicate that the related statements about limits hold for $x \to a, x \to a^+, x \to a^-, x \to \infty$ and $x \to -\infty$.
That is, $\lim f(x)$ is understood to be any of the limits listed below.

- $\lim\limits_{x\to a^+} f(x)$,
- $\lim\limits_{x\to a^-} f(x)$,
- $\lim\limits_{x\to a} f(x)$,
- $\lim\limits_{x\to \infty} f(x)$,
- $\lim\limits_{x\to -\infty} f(x)$.

Similarly, we use the expression $\lim\limits_{x\uparrow a} f(x)$ to indicate that the related statements about limits hold for $x \to a, x \to a^+$ and $x \to a^-$.
That is, $\lim\limits_{x\uparrow a} f(x)$ is understood to be any of the limits listed below.

- $\lim\limits_{x\to a^+} f(x)$,
- $\lim\limits_{x\to a^-} f(x)$,
- $\lim\limits_{x\to a} f(x)$.

Note. In all limit statements, coloured expressions $\lim\ f(x)$ or $\lim\ _{x\uparrow a} f(x)$ are taken as limits of the *same* type.

In the proofs presented in this chapter, we must keep in mind the inequalities 1-6 (page xii) listed in the Introduction together with the following three facts.

1. If a property

$$P_1 \quad \text{holds for every number } |u| < \delta_1$$

and a property

$$P_2 \quad \text{holds for every number } |v| < \delta_2,$$

then for every number $|x| < \delta$, where $\delta = \min(\delta_1, \delta_2)$

$$|x| < \delta \leq \delta_1 \quad \text{and} \quad |x| < \delta \leq \delta_2.$$

Therefore, both properties P_1 and P_2 hold for every $|x| < \delta$.

2. If a property

$$P_1 \quad \text{holds for every number } u > V_1$$

and a property

$$P_2 \quad \text{holds for every number } v > V_2,$$

then for every number $x > V$, where $V = \max(V_1, V_2)$

$$x > V \geq V_1 \quad \text{and} \quad x > V \geq V_2.$$

Therefore, both properties P_1 and P_2 hold for every $x > V$.

3. If a property

$$P_1 \quad \text{holds for every number } u < U_1$$

and a property

$$P_2 \quad \text{holds for every number } v < U_2,$$

then for every number $x < U$, where $U = \min(U_1, U_2)$

$$x < U \leq U_1 \quad \text{and} \quad x < U \leq U_2.$$

Therefore, both properties P_1 and P_2 hold for every $x < U$.

In the next proposition, we prove the uniqueness of the finite limits.
It is important to teach this very valuable property of limits.

Proposition 3.1. *If*

$$\lim f(x) = M_1 \quad \textit{and} \quad \lim f(x) = M_2,$$

then $M_1 = M_2$.

Proof. We present the proof for $x \to a$ and $x \to \infty$. The proofs for $x \to a^+$, $x \to a^-$ and $x \to -\infty$ are left as exercises.

Let $\varepsilon > 0$ be any positive number.

For the positive number $\frac{\varepsilon}{2} > 0$, there are positive numbers $\delta_1, \delta_2 > 0$, such that

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad 0 < |x-a| < \delta_1$$

and

$$|f(x) - M_2| < \frac{\varepsilon}{2} \quad \text{for every} \quad 0 < |x-a| < \delta_2.$$

These two statements hold for $\delta = \min(\delta_1, \delta_2) > 0$ (see fact 1 above). Thus, for any number $0 < |x-a| < \delta$, by the inequality 2 (page xii)

$$|M_1 - M_2| = |f(x) - M_2 + M_1 - f(x)| \le |f(x) - M_1| + |f(x) - M_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is any positive number, by Exercise 15 on page 73, we conclude that $M_1 = M_2$.

The proof for $x \to \infty$ is similar.
For the positive number $\frac{\varepsilon}{2} > 0$, there are positive numbers $V_1 > 0$ and $V_2 > 0$ such that

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad x > V_1$$

and

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad x > V_2.$$

These two previous statements hold for $V = \max(V_1, V_2) > 0$ (see fact 2 above). Thus, as above, for any number $x > V$

$$|M_1 - M_2| = |f(x) - M_2 + M_1 - f(x)| \le |f(x) - M_1| + |f(x) - M_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Again, since ε is any positive number, we conclude that $M_1 = M_2$. Q.E.D.

Laws of Limits

It must be strongly emphasized that the Laws of Limits applies only to *finite* limits. The incorrect application of these laws creates all kinds of misunderstandings.

Theorem 3.2. ***Laws of Limits.*** *If*

$$\lim f(x) = K \qquad \textit{and} \qquad \lim g(x) = M,$$

then

a. $\lim\, [f(x) \pm g(x)] = K \pm M$

b. $\lim\, [f(x)g(x)] = KM$

c. $\lim\, [cf(x)] = cK$ *for any constant* c*.*

d. $\lim \dfrac{1}{g(x)} = \dfrac{1}{M}$ *only if* $M \neq 0$*.*

e. $\lim \dfrac{f(x)}{g(x)} = \dfrac{K}{M}$ *only if* $M \neq 0$*.*

Proof. We present the proof for $x \to a$ and $x \to \infty$. The proofs of *some* of the remaining cases are left as exercises.

For $x \to a$, we apply Definition 1.11 (page 22).
Let $\varepsilon > 0$ be any positive number.

a. For the positive number $\dfrac{\varepsilon}{2} > 0$ there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$|f(x) - K| < \frac{\varepsilon}{2} \qquad \text{for every} \qquad 0 < |x - a| < \delta_1, \tag{3.1}$$

and

$$|g(x) - M| < \frac{\varepsilon}{2} \qquad \text{for every} \qquad 0 < |x - a| < \delta_2. \tag{3.2}$$

By the fact 1 above, both statements (3.1) and (3.2) hold for

$$0 < |x - a| < \delta \quad \text{where } \delta = \min(\delta_1, \delta_2) > 0.$$

Thus, for every $0 < |x - a| < \delta$

$$|(f(x) \pm g(x)) - (K \pm M)| \leq |f(x) - K| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

b. We consider three cases.
Case 1: $K = M = 0$. For the positive number $\sqrt{\varepsilon} > 0$ there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$|f(x)| < \sqrt{\varepsilon} \qquad \text{for every} \qquad 0 < |x-a| < \delta_1, \tag{3.3}$$

and

$$|g(x)| < \sqrt{\varepsilon} \qquad \text{for every} \qquad 0 < |x-a| < \delta_2. \tag{3.4}$$

By the fact 1 above, both statements (3.3) and (3.4) hold for

$$0 < |x-a| < \delta \quad \text{where } \delta = \min(\delta_1, \delta_2) > 0.$$

Thus, for every $0 < |x-a| < \delta$

$$|f(x)g(x)| = |f(x)||g(x)| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon.$$

Case 2: $K = 0$ and $M \neq 0$. For the positive number

$$\frac{\varepsilon}{\varepsilon + |M|} > 0 \quad \text{there are positive numbers } \delta_1, \delta_2 > 0,$$

such that

$$|f(x)| < \frac{\varepsilon}{\varepsilon + |M|} \qquad \text{for every} \qquad 0 < |x-a| < \delta_1, \tag{3.5}$$

and

$$|g(x)| - |M| \leq |g(x) - M| < \varepsilon \qquad \text{for every} \qquad 0 < |x-a| < \delta_2. \tag{3.6}$$

By the fact 1 above, both statements (3.5) and (3.6) hold for

$$0 < |x-a| < \delta \quad \text{where } \delta = \min(\delta_1, \delta_2) > 0.$$

Thus, for every $0 < |x-a| < \delta$

$$|f(x)g(x)| = |f(x)||g(x)| < \left[\frac{\varepsilon}{\varepsilon + |M|}\right](\varepsilon + M) = \frac{\varepsilon(\varepsilon + |M|)}{\varepsilon + |M|} = \varepsilon.$$

Case 3: $K, M \neq 0$. For the positive number

$$\frac{\varepsilon}{2|M|} > 0, \quad \text{there is a positive number } \delta_1 > 0,$$

such that

$$|f(x)| - |K| \leq |f(x) - K| < \frac{\varepsilon}{2|M|} \qquad \text{for every} \qquad 0 < |x-a| < \delta_1. \tag{3.7}$$

Then,

$$|f(x)| < \frac{\varepsilon}{2|M|} + |K| = \frac{\varepsilon + 2|KM|}{2|M|} \qquad \text{for every} \qquad 0 < |x-a| < \delta_1. \quad (3.8)$$

For the positive number

$$\frac{|M|\varepsilon}{\varepsilon + 2|KM|)} > 0, \quad \text{there is a positive number } \delta_2 > 0,$$

such that

$$|g(x) - M| < \frac{|M|\varepsilon}{\varepsilon + 2|KM|)} \qquad \text{for every} \qquad 0 < |x-a| < \delta_2. \qquad (3.9)$$

By the fact 1 above, the statements (3.7), (3.8) and (3.9) hold for

$$0 < |x-a| < \delta \quad \text{where } \delta = \min(\delta_1, \delta_2) > 0.$$

Thus, for every $0 < |x-a| < \delta$

$$\begin{aligned}
|f(x)g(x) - KM| &\leq |f(x) - f(x)M + f(x)M - KM| \\
&< |f(x)||g(x) - M| + |M||f(x) - K| \\
&< \left(\frac{\varepsilon + 2|KM|}{2|M|}\right)\left(\frac{|M|\varepsilon}{\varepsilon + 2|KM|)}\right) + \frac{|M|\varepsilon}{2|M|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

In either case the statement (1.15) on page 22 holds.

c. By Exercise 1 (page 71),

$$\lim_{x \to a} c = c \quad \text{for any constant } c.$$

By part (b)

$$cK = \left(\lim_{x \to a} c\right)\left(\lim_{x \to a} f(x)\right) = \lim_{x \to a}[cf(x)].$$

d. For the positive number

$$\frac{\varepsilon|M|^2}{1 + |M|\varepsilon} > 0, \quad \text{there is a positive number } \delta > 0,$$

such that

$$|g(x) - M| < \frac{\varepsilon|M|^2}{1 + |M|\varepsilon} \qquad \text{for every} \qquad 0 < |x-a| < \delta. \qquad (3.10)$$

By inequality 3 (page xii)

$$|M| - |g(x)| \leq |g(x) - M| < \frac{\varepsilon|M|^2}{1 + |M|\varepsilon},$$

and we have that

$$-|g(x)| < \frac{\varepsilon|M|^2}{1 + |M|\varepsilon} - |M| = -\frac{|M|}{1 + |M|\varepsilon}.$$

By properties 5 and 6 of inequalities (page xii)

$$\frac{1}{|g(x)|} < \frac{1 + |M|\varepsilon}{|M|} \qquad \text{for every} \qquad 0 < |x - a| < \delta. \tag{3.11}$$

Thus, for every $0 < |x - a| < \delta$ the statements (3.10) and (3.11) hold and

$$\begin{aligned} \left|\frac{1}{g(x)} - \frac{1}{M}\right| &= \left|\frac{M - g(x)}{Mg(x)}\right| \\ &< \left(\frac{\varepsilon|M|^2}{1 + |M|\varepsilon}\right)\left(\frac{1 + |M|\varepsilon}{|M|^2}\right) = \varepsilon. \end{aligned}$$

e. This follows from parts (c) and (d).

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} \frac{1}{g(x)}\right) = K\left(\frac{1}{M}\right) = \frac{K}{M}.$$

For the proof of $x \to \infty$, we apply Definition 2.5 (page 35).
Let $\varepsilon > 0$ be any positive number.

a. For the positive number $\frac{\varepsilon}{2} > 0$, there are positive numbers $M_1, M_2 > 0$, such that

$$|f(x) - K| < \frac{\varepsilon}{2} \qquad \text{for every} \qquad x > M_1, \tag{3.12}$$

and

$$|g(x) - L| < \frac{\varepsilon}{2} \qquad \text{for every} \qquad x > M_2. \tag{3.13}$$

By the fact 2 above, both statements (3.12) and (3.13) hold for every

$$x > M \quad \text{where } M = \max(M_1, M_2) > 0.$$

Thus, for every $x > M$

$$|(f(x) \pm g(x)) - (K \pm L)| \leq |f(x) - K| + |g(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

b. We consider the same three cases.

Case 1: $K = L = 0$. For the positive number $\sqrt{\varepsilon} > 0$ there are positive numbers $M_1, M_2 > 0$ such that

$$|f(x)| < \sqrt{\varepsilon} \qquad \text{for every} \qquad x > M_1, \tag{3.14}$$

and

$$|g(x)| < \sqrt{\varepsilon} \qquad \text{for every} \qquad x > M_2. \tag{3.15}$$

By the fact 2 above, both statements (3.14) and (3.15) hold for every number

$$x > M \quad \text{where } M = \max(M_1, M_2) > 0.$$

Thus, for every $x > M$

$$|f(x)g(x)| < \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon \qquad \text{for every} \qquad x > M.$$

Case 2: $K = 0$ and $L \neq 0$. For the positive number

$$\frac{\varepsilon}{\varepsilon + |L|} \quad \text{there are positive numbers } M_1, M_2 > 0,$$

such that

$$|f(x)| < \frac{\varepsilon}{\varepsilon + |L|} \qquad \text{for every} \qquad x > M_1, \tag{3.16}$$

and

$$|g(x)| - |L| \leq |g(x) - L| < \varepsilon \qquad \text{for every} \qquad x > M_2. \tag{3.17}$$

By the fact 2 above, both statements (3.16) and (3.17) hold for every number

$$x > M \quad \text{where } M = \max(M_1, M_2) > 0.$$

Thus, for every $x > M$

$$|f(x)g(x)| < \frac{\varepsilon(\varepsilon + |L|)}{\varepsilon + |L|} = \varepsilon.$$

Case 3: $K, L \neq 0$. For the positive number

$$\frac{\varepsilon}{2|L|} > 0, \quad \text{there is a positive number } M_1 > 0,$$

such that

$$|f(x)| - |K| \leq |f(x) - K| < \frac{\varepsilon}{2|M|} \qquad \text{for every} \qquad x > M_1. \tag{3.18}$$

Then,

$$|f(x)| < \frac{\varepsilon}{2|L|} + |K| = \frac{\varepsilon + 2|KL|}{2|L|} \qquad \text{for every} \qquad x > M_1. \tag{3.19}$$

For the positive number

$$\frac{|L|\varepsilon}{\varepsilon + 2|KL|)} > 0, \quad \text{there is a positive number } M_2 > 0,$$

such that

$$|g(x) - L| < \frac{|L|\varepsilon}{\varepsilon + 2|KL|)} \qquad \text{for every} \qquad x > M_2. \tag{3.20}$$

By the fact 2 above, the statements (3.18), (3.19) and (3.20) hold for

$$x > M \quad \text{where } M = \max(M_1, M_2) > 0.$$

Thus, for every $x > M$

$$\begin{aligned}
|f(x)g(x) - KL| &\leq |f(x) - f(x)L + f(x)L - KL| \\
&< |f(x)||g(x) - L| + |L||f(x) - K| \\
&< \left(\frac{\varepsilon + 2|KL|}{2|L|}\right)\left(\frac{|L|\varepsilon}{\varepsilon + 2|KL|)}\right) + \frac{|L|\varepsilon}{2|L|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

In any case the statement (2.4) on page 35 holds.

c. By Exercise 1 (page 71),

$$\lim_{x\to\infty} c = c \quad \text{for any constant } c.$$

By part (b)

$$cK = \left(\lim_{x\to\infty} c\right)\left(\lim_{x\to\infty} f(x)\right) = \lim_{x\to\infty}[cf(x)].$$

d. For the positive number $\dfrac{\varepsilon|L|^2}{1 + |L|\varepsilon} > 0$, there is a positive number $M > 0$ such that

$$|g(x) - L| < \frac{\varepsilon|L|^2}{1 + |L|\varepsilon} \qquad \text{for every} \qquad x > M. \tag{3.21}$$

By inequality 3 (page xii)

$$|M| - |g(x)| \le |g(x) - L| < \frac{\varepsilon |L|^2}{1 + |L|\varepsilon},$$

we have that

$$-|g(x)| < \frac{\varepsilon |L|^2}{1 + |L|\varepsilon} - |L| = -\frac{|L|}{1 + |L|\varepsilon}.$$

By properties 5 and 6 of inequalities (page xii)

$$\frac{1}{|g(x)|} < \frac{1 + |L|\varepsilon}{|L|} \qquad \text{for every} \qquad x > M. \tag{3.22}$$

Thus, for every $x > M$ the statement (3.22) holds and

$$\begin{aligned} \left| \frac{1}{g(x)} - \frac{1}{L} \right| &= \left| \frac{L - g(x)}{L(g(x))} \right| \\ &< \left(\frac{\varepsilon |L|^2}{1 + |L|\varepsilon} \right) \left(\frac{1 + |L|\varepsilon}{|L|^2} \right) = \varepsilon. \end{aligned}$$

e. This follows from parts (c) and (d).

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \left(\lim_{x\to\infty} f(x) \right) \left(\lim_{x\to\infty} \frac{1}{g(x)} \right) = K \left(\frac{1}{L} \right) = \frac{K}{L}.$$

Q.E.D.

Part (a) of the Laws of Limits is interpreted as “the limit of the sum is the sum of the limits.” That is

$$\lim\ (f + g) = \lim\ f(x) + \lim\ g(x).$$

Similarly for the product and quotient of functions.

The application of the Laws of Limits to the limit

$$\lim_{x\to a} x = a \quad \text{for every number } a. \tag{3.23}$$

yields the following corollary (see Exercise 4 on page 71).

Corollary 3.3. *If $P(x)$ is a polynomial function and $R(x) = \dfrac{p(x)}{q(x)}$ is a rational function, then for any number a*

a. $\lim_{x \uparrow a^+} P(x) = P(a)$

b. $\lim_{x \uparrow a^+} \frac{p(x)}{q(x)} = R(a) \quad$ *if* $\quad q(a) \neq 0.$

Proof. The proof of the limit

$$\lim_{x \to a} x^n = a^n \quad \text{is by induction.}$$

The limit (3.23) is for $n = 1$.
If

$$\lim_{x \to a} x^n = a^n,$$

then by part (b) of Theorem 3.2 (page 47)

$$\lim_{x \to a} x^{n+1} = [\lim_{x \to a} x^n][\lim_{x \to a} x] = a^n(a) = a^{n+1}.$$

Thus, this proof is a direct consequence of parts (a),(b) and (e) of Theorem 3.2 (page 47). Q.E.D.

We must provide counterexamples whenever possible to convince students of the importance of applying results correctly.

We show in the next example that the converse of part (b) of the Laws of Limits is not true.

Example 3.1. The limit,

$$\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0 \quad \text{exist.}$$

However,

$$\lim_{x \to 0^+} x = 0 \qquad \text{and} \qquad \lim_{x \to 0^+} \sin\left(\frac{1}{x}\right) \quad \text{does not exist.}$$

See Example 1.9 (page 17). □

Finite Limits from the Right and Left

From Figures 1.7 to 1.9 (pages 13 to 14), we see that

$f(x) \to R \quad \text{as } x \to a^+,$

and we read

$f(x)$ tends to R as x tends to a from the right.

Moreover, from Figure 1.7 (page 13), we have

$f(x) > R \quad \text{for } x \to a^+$

and

$f(x)$ tends to R from the *right* as x tends to a from the right.

We write

$$f(x) \to R^+ \qquad \text{as} \qquad x \to a^+.$$

Similarly, from Figure 1.8 (page 14), we have

$f(x) < R \quad \text{for } x \to a^+$

and

$f(x)$ tends to R from the *left* as x tends to a from the right.

We write

$$f(x) \to R^- \qquad \text{as} \qquad x \to a^+.$$

From Figure 1.9 (page 14), we see that $f(x)$ tends to R from both sides, right and left, as x tends to a from the right.

Also, from Figures 2.2 to 2.4 (pages 34 to 34), we see that

$f(x) \to K \quad \text{as } x \to \infty,$

and we read

> $f(x)$ tends to K as x tends to infinity (increases without bounds).

From Figure 2.2 (page 34), we have

$$f(x) < K \quad \text{for } x \to \infty$$

and

> $f(x)$ tends to K from the *left* as x tends to infinity (increases without bounds).

We write

$$f(x) \to K^- \qquad \text{as} \qquad x \to \infty.$$

Similarly, from Figure 2.3 (page 34), we have

$$f(x) > K \quad \text{for } x \to \infty$$

and

> $f(x)$ tends to K from the *right* as x tends to infinity (increases without bounds).

We write

$$f(x) \to K^+ \qquad \text{as} \qquad x \to \infty.$$

From Figure 2.4 (page 34), the function $f(x)$ tends to K from both sides, right and left, as x tends to infinity.

These intuitive ideas are defined next.

Definition 3.4. a. The *limit at* a *from the right of a function* g *is* R *from the right* if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$0 < f(x) - R < \varepsilon \qquad \text{for every} \quad 0 < x - a < \delta.$$

We write either

$$\lim_{x \to a^+} f(x) = R^+ \qquad \text{or} \qquad g(x) \to R^+ \quad \text{as} \quad x \to a^+$$

and read

$g(x)$ tends to R from the right as x tends to a from the right.

Figure A.6 (page 265) shows the graphic representation of $\lim_{x \to a^+} f(x) = R^+$.

b. The *limit at a from the left of a function f is L from the left* if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$0 < L - f(x) < \varepsilon \qquad \text{for every} \quad 0 < a - x < \delta.$$

We write either

$$\lim_{x \to a^-} f(x) = L^- \qquad \text{or} \qquad f(x) \to L^- \quad \text{as} \quad x \to a^-$$

and read

$f(x)$ tends to L from the left as x tends to a from the left.

Figure A.12 (page 268) shows the graphic representation of $\lim_{x \to a^-} f(x) = L^-$.

c. The *limit at a of a function f is R from the right* if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$0 < f(x) - R < \varepsilon \qquad \text{for every} \quad 0 < |x - a| < \delta.$$

We write either

$$\lim_{x \to a} f(x) = R^+ \qquad \text{or} \qquad f(x) \to R^+ \quad \text{as} \quad x \to a$$

and read

$f(x)$ tends to R from the right as x tends to a.

Figure A.8 (page 266) shows the graphic representation of $\lim_{x \to a} f(x) = R^+$.

d. The *limit at a of a function f is L from the left* if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$0 < L - f(x) < \varepsilon \qquad \text{for every} \quad 0 < |x - a| < \delta.$$

We write either

$$\lim_{x \to a} f(x) = L^- \qquad \text{or} \qquad f(x) \to L^- \quad \text{as} \quad x \to a$$

and read

> $f(x)$ tends to L from the left as x tends to a.

Figure A.13 (page 268) shows the graphic representation of $\lim\limits_{x \to a} f(x) = L^-$.

e. The *limit at infinity of a function f is R from the right* if for any positive number $\varepsilon > 0$ there is a positive number $V > 0$ such that

$$0 < f(x) - R < \varepsilon \qquad \text{for every} \quad x > V.$$

We write either

$$\lim_{x \to \infty} f(x) = R^+ \qquad \text{or} \qquad f(x) \to R^+ \quad \text{as} \quad x \to \infty$$

and read

> $f(x)$ tends to R from the right as x tends to infinity (increases without bound).

Figure A.9 (page 266) shows the graphic representation of $\lim\limits_{x \to \infty} f(x) = R^+$.

f. The *limit at negative infinity of a function f is R from the right* if for any positive number $\varepsilon > 0$ there is a negative number $U < 0$ such that

$$0 < f(x) - R < \varepsilon \qquad \text{for every} \quad x < U.$$

We write either

$$\lim_{x \to -\infty} f(x) = R^+ \qquad \text{or} \qquad f(x) \to R^+ \quad \text{as} \quad x \to -\infty$$

and read

> $f(x)$ tends to R from the right as x tends to negative infinity (decreases without bound).

Figure A.10 (page 267) shows the graphic representation of $\lim\limits_{x \to -\infty} f(x) = R^+$.

Remark 3.5. *1. Definition 3.4 does not cover all possible similar cases. See Exercise 7 (page 71).*

2. The notation in Definition 3.4 is not standard but it is self-evident, and we apply it in Theorem 3.7.

Example 3.2. We know that

$$\lim_{x \to 0} \cos x = 1.$$

Hence, for any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|\cos x - 1| < \varepsilon \quad \text{for every} \quad 0 < |x| < \delta_1.$$

Let $\delta = \min\left(\delta_1, \dfrac{\pi}{2}\right) > 0$. Thus,

$$0 \leq 1 - \cos x < \varepsilon \quad \text{for every} \quad 0 < |x| < \delta.$$

Therefore

$$\lim_{x \to 0} \cos x = 1^-.$$

See the graph of the cosine function in Figure 1.4 (page 6). □

Example 3.3. We know that

$$\lim_{x \to 0} \sin x = 0.$$

Hence, for any positive number $\varepsilon > 0$, there is a positive number $\delta_1 > 0$, such that

$$|\sin x| < \varepsilon \quad \text{for every} \quad 0 < |x| < \delta_1.$$

If $\delta = \min(\delta_1, \pi) > 0$, then

$$0 < \sin x < \varepsilon \quad \text{for} \quad 0 < x < \delta.$$

Thus, $\displaystyle\lim_{x \to 0^+} \sin x = 0^+$.
Also,

$$0 < -\sin x = \sin(-x) < \varepsilon \quad \text{for} \quad 0 < -x < \delta.$$

Thus, $\displaystyle\lim_{x \to 0^-} \sin x = 0^-$. □

In the next example we show that if $\displaystyle\lim_{x \to a} g(x) = M$, then it is not true that either $g(x) \to M^+$ or $g(x) \to M^-$ as $x \to a$.

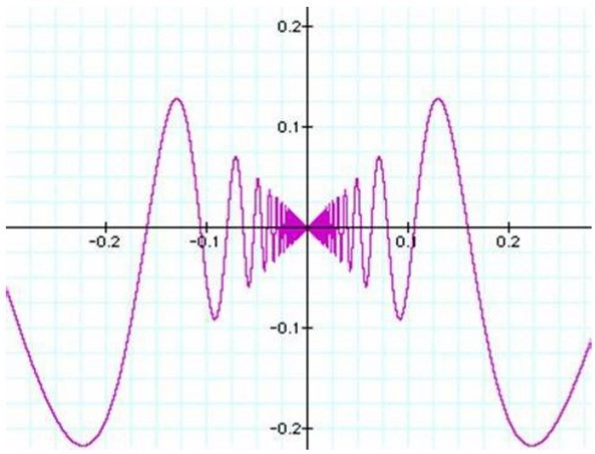


Figure 3.1: Graph of the function $h(x) = x \sin\left(\dfrac{1}{x}\right)$

Example 3.4. Figure 3.1 shows the graph of the function $h(x) = x \sin\left(\dfrac{1}{x}\right)$.

We will see in the next section that the limit of this function at zero is zero.

Hence, for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|h(x)| < \varepsilon \qquad \text{for every} \qquad 0 < |x| < \delta.$$

By Example 1.9 (page 17), for this particular $\delta > 0$, there are integers k and n such that

$$0 < \frac{2}{(1+4k)\pi} < \delta \quad \text{and} \quad 0 < \frac{2}{(4n+3)\pi} < \delta.$$

Let $a = \dfrac{2}{(1+4k)\pi}$ and $b = \dfrac{2}{(4n+3)\pi}$. Thus,

$$h(a) = \frac{(1+4k)\pi}{2} > 0 \quad \text{and} \quad h(b) = -\frac{(4n+3)\pi}{2} < 0.$$

Hence,

$$h(x) \ngeq 0 \quad \text{nor} \quad -h(x) \ngeq 0 \quad \text{for every } 0 < x < \delta.$$

Therefore, parts (a) and (b) of Definition 3.4 (page 56) do not hold and

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0 \quad \text{from neither right nor left.}$$

$\square$

The next proposition establishes under which conditions the limits in Definition 3.4 (page 56) are from the right or left.

Proposition 3.6. *a. If*

- $\lim_{x \to a} g(x) = M$ *and*
- $g(x) > M$ *for* $x \to a$,

then

$$\lim_{x \to a} g(x) = M^{+}.$$

b. If

- $\lim_{x \to a} g(x) = M$ *and*
- $g(x) < M$ *for* $x \to a$,

then

$$\lim_{x \to a} g(x) = M^{-}.$$

By Definition 1.1 (page 1), this proposition also holds for $x \to a^{\pm}$ and $x \to \pm\infty$.

Proof. a. For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|M - g(x)| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta_1.$$

Also, there is a positive number $\nu > 0$ such that

$$g(x) - M > 0 \quad \text{for every } 0 < |x - a| < \nu.$$

Let $\delta = \min(\nu, \delta_1) > 0$. Then

$$0 < g(x) - M = |g(x) - M| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta.$$

b. For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|M - g(x)| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta_1.$$

Also, there is a positive number $\nu > 0$ such that

$$M - g(x) > 0 \quad \text{for every } 0 < |x - a| < \nu.$$

Let $\delta = \min(\nu, \delta_1) > 0$. Then

$$0 < M - g(x) = |M - g(x)| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta.$$

Q.E.D.

Example 3.5. By Example 2.8 (page 37) and Example 2.12 (page 42)

$$\lim_{x\to\infty} \frac{1}{x} = 0 = \lim_{x\to-\infty} \frac{1}{x}.$$

Since $\frac{1}{x} > 0$ for $x \to 0^+$ and $\frac{1}{x} < 0$ for $x \to 0^-$, we have that,

$$\lim_{x\to\infty} \frac{1}{x} = 0^+ \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0^-.$$

□

Finite Limits of Composition of Functions

We show, once more, in the next examples the importance that the domain of functions has in determining whether a limit is well defined or not.

Example 3.6. The domain of the composition

$$f(x) = \sqrt{\sin x}$$

of the sine and square root functions consists of the union of all intervals where $\sin x \geq 0$, such as $[0, \pi]$.
Note that the interval $(-\pi, 0)$ and infinite intervals are not contained in the domain.
Hence,

a. $\lim_{x\to 0^+} \sqrt{\sin x}$ is defined.

b. $\lim_{x\to 0^-} \sqrt{\sin x}$ is undefined.

c. $\lim_{x\to\infty} \sqrt{\sin x}$ is undefined.

d. $\lim_{x\to-\infty} \sqrt{\sin x}$ is undefined.

□

Example 3.7. The domain of the function

$$g(x) = \sqrt{\ln x} \quad \text{is the interval } [1, \infty).$$

Hence,

$$\lim_{x\to 1^+} \sqrt{\ln x} \quad \text{is defined} \quad \text{but} \quad \lim_{x\to 1^-} \sqrt{\ln x} \quad \text{is undefined.}$$

On the other hand

$$\lim_{x\to \infty} \sqrt{\ln x} \quad \text{is defined} \quad \text{but} \quad \lim_{x\to -\infty} \sqrt{\ln x} \quad \text{is undefined.}$$

$\square$

We apply the next theorem to evaluate limits of composition of functions.

Theorem 3.7. *a. If* $\lim g(x) = b^+$ *and* $\lim\limits_{y\to b^+} f(y) = c$*, then*

$$\lim f(g(x)) = c.$$

b. If $\lim g(x) = b^-$ *and* $\lim\limits_{y\to b^-} f(y) = c$*, then*

$$\lim f(g(x)) = c.$$

c. If $\lim g(x) = b$ *and* $\lim\limits_{y\to b} f(y) = c$*, then*

$$\lim f(g(x)) = c.$$

Proof. We give the proofs of part (a) for the limits

$$\lim_{x\to a} g(x) = b^+ \quad \text{and} \quad \lim_{x\to\infty} g(x) = b^+.$$

The remaining cases are left as exercises.

Let $\lim\limits_{x\to a} g(x) = b^+$ and $\lim\limits_{y\to b^+} f(y) = c$.

For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|f(y) - c| < \varepsilon \quad \text{for every} \quad 0 < y - b < \delta_1.$$

For this $\delta_1 > 0$ there is a positive number $\delta_2 > 0$ such that

$$0 < g(x) - b < \delta_1 \quad \text{for every} \quad 0 < |x - a| < \delta_2.$$

Thus,

$$|f(g(x)) - c| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta_1.$$

Let $\quad \lim_{x\to\infty} g(x) = b^+ \qquad$ and $\qquad \lim_{y\to b^+} f(y) = c.$

For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|f(y) - c| < \varepsilon \quad \text{for every} \quad 0 < y - b < \delta_1.$$

For this $\delta_1 > 0$ there is a positive number $V > 0$ such that

$$0 < g(x) - b < \delta_1 \quad \text{for every} \quad x > V.$$

Thus,

$$|f(g(x)) - c| < \varepsilon \quad \text{for every} \quad x > V.$$

Q.E.D.

A particular case of Theorem 3.7 can be stated as follows.

If $y = g(x) \to b^+$ as $x \to a^+$ and $f(y) \to c$ as $y \to b^+$, then

$$\lim_{x\to a^+} f(g(x)) = \lim_{y\to b^+} f(y) = c.$$

It is incorrect to state this conclusion as

$$\lim_{x\to a^+} f(g(x)) = f\left(\lim_{x\to a^+} g(x)\right) = f(b) = c, \tag{3.24}$$

because the function f may not be defined at b. We can conclude in this manner, only when the function f is continuous at b from the right, as shown in the next example.

Example 3.8. Let f be the function

$$f(y) = \begin{cases} 1 & \text{if} \quad y \geq 1 \\ 0 & \text{if} \quad 0 < y < 1 \\ -1 & \text{if} \quad y \leq 0 \end{cases}$$

The domain of this function is all real numbers.

We apply Theorem 3.7 (page 63), to determine whether the limit

$$\lim_{x\to 0} f(\cos x) \quad \text{exists.}$$

The domain of $f(\cos x)$ is all real numbers, hence the limit we are considering is well defined. By Example 3.2 (page 59)

$$\lim_{x\to 0} \cos x = 1^-. \tag{3.25}$$

By definition of f

$$\lim_{y\to 1^-} f(y) = 0. \tag{3.26}$$

If $y = \cos x$, then by part (b) of Theorem 3.7 (page 63) and (3.26) and (3.25)

$$\lim_{x\to 0} f(\cos x) = \lim_{y\to 1^-} f(y) = 0 \neq f(1) = 1.$$

We apply the same theorem to determine whether the limit

$$\lim_{x\to \pi/2} f(\cos x) \quad \text{exists.}$$

We know that $\lim\limits_{x\to\pi/2} \cos x = 0$ but

$$\lim_{x\to\pi/2^+} \cos x = 0^- \qquad \text{and} \qquad \lim_{x\to\pi/2^-} \cos x = 0^+. \tag{3.27}$$

By definition of f

$$\lim_{y\to 0^-} f(y) = -1 \qquad \text{and} \qquad \lim_{y\to 0^+} f(y) = 0.$$

If $y = \cos x$, then by part (b) of Theorem 3.7 and (3.27)

$$\lim_{x\to\pi/2^+} f(\cos x) = \lim_{y\to 0^-} f(y) = -1$$

and by part (a) of Theorem 3.7 and (3.27)

$$\lim_{x\to\pi/2^-} f(\cos x) = \lim_{y\to 0^+} f(y) = 0$$

Therefore by part (c) of Proposition 1.12 (page 24) the limit

$$\lim_{x\to\pi/2} f(\cos x) \quad \text{does not exist.}$$

This shows that the evaluation of the limit shown below is incorrect.

$$\lim_{x\to\pi/2} f(\cos x) = f\left(\lim_{x\to\pi/2} \cos x\right) = f(0) = -1.$$

$\square$

Theorem 3.7 (page 63) and Proposition 3.6 (page 61) are precisely the results we must apply to functions of the form $\sqrt[n]{g(x)}$ and $\dfrac{1}{\sqrt[n]{g(x)}}$, for some positive integer n.

Corollary 3.8. *Let n be a positive integer. If*

- *the function $\sqrt[n]{g(x)}$ is defined for $x \to a$,*
- $\lim\limits_{x\to a} g(x) = b$*, and*
- $\sqrt[n]{b}$ *is defined,*

then

$$\lim_{x\to a} \sqrt[n]{g(x)} = \sqrt[n]{b}.$$

Corollary 3.9. *Let n be a positive integer. If*

- *the function $\sqrt[n]{g(x)}$ is defined for $x \to a$,*
- $\lim\limits_{x\to a} g(x) = b$*, and*
- $\sqrt[n]{b}$ *is defined and non equal to 0,*

then

$$\lim_{x\to a} \frac{1}{\sqrt[n]{g(x)}} = \frac{1}{\sqrt[n]{b}}.$$

These two corollaries also hold for $x \to a^{\pm}$ and $x \to \pm\infty$.

We apply these corollaries in the next two examples.

Example 3.9. The composition $h(x) = f(g(x))$ where

$$f(x) = x - 3 \quad \text{and} \quad g(x) = \sqrt{x}$$

is defined for $x \to 3^{+}$. In order to apply Corollary 3.8 we must consider the limit

$$\lim_{x\to 3^{+}} \sqrt{x-3} = 0.$$

Since $x - 3 \geq 0$ for $x \to 3^+$, by part (a) of Proposition 3.6 (page 61)

$$\lim_{x \to 3^+} \sqrt{x - 3} = 0^+.$$

Applying Corollary 3.8 (page 66) with $n = 2$ and $t = x - 3$, we have

$$\lim_{x \to 3^+} \sqrt{x - 3} = \lim_{t \to 0^+} \sqrt{t} = 0.$$

$\square$

Example 3.10. The limit

$$\lim_{x \to 0} \frac{1}{\sqrt{\cos x}} \quad \text{is defined.}$$

If $t = \cos x$, then $t \to 1^-$ as $x \to 0$, by Example 3.2 (page 59). Thus,

$$\lim_{x \to 0} \frac{1}{\sqrt{\cos x}} = \lim_{t \to 1^-} \frac{1}{\sqrt{t}} = 1.$$

$\square$

Example 3.11. We have the limits

$$\lim_{x \to 0} 1 - x = 1 \quad \text{and} \quad \lim_{x \to 1^+} \sqrt{\ln x} = 0.$$

Let $f(x) = 1 - x$ and $g(x) = \sqrt{\ln x}$. The composition

$$g(f(x)) = \sqrt{\ln(1 - x)} \quad \text{is defined for } x \to 1^-,$$

because its domain is $(-\infty, 1)$.

Hence, Theorem 3.7 (page 63) applies to the function

$$g(f(x)) = \sqrt{\ln(1 - x)} \quad \text{for } x \to 0^-.$$

Indeed, if $t = 1 - x$, then $t \to 1^+$ as $x \to 0^-$, and

$$\lim_{x \to 0^-} \sqrt{\ln(1 - x)} = \lim_{t \to 1^+} \sqrt{\ln t} = 0.$$

$\square$

Limits and Inequalities: The Squeeze Theorem

For any two functions $f(x)$ and $g(x)$, we apply Definition 1.1, the statements (2.1) and (2.2) on pages 30 and 32 respectively, with the property P as either

$$f(x) \leq g(x) \quad \text{or} \quad f(x) < g(x).$$

For inequalities of functions we have the following three important results.

Theorem 3.10. *If*

$$f(x) \leq 0 \quad \textit{for } x \to a$$

and

$$\lim_{x\to a} f(x) = L,$$

then $L \leq 0$.

This theorem also holds for $x \to a^{\pm}$ and $x \to \pm\infty$.

Proof. There is a positive number $\mu > 0$, such that

$$f(x) \leq 0 \quad \text{for every} \quad 0 < |x - a| < \mu. \tag{3.28}$$

For any positive number $\varepsilon > 0$, there is a positive number δ_1 such that

$$|f(x) - L| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta_1. \tag{3.29}$$

Both statements (3.28) and (3.29) hold for every $0 < |x - a| < \delta$ where $\delta = \min(\mu, \delta_1) > 0$. Thus,

$$-\varepsilon < L - f(x) < \varepsilon \quad \text{and} \quad f(x) < 0.$$

Therefore,

$$L = L - f(x) + f(x) \leq L - f(x) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, L cannot be positive. Hence, $L \leq 0$. Q.E.D.

The following two corollaries are direct consequences of this theorem.

Corollary 3.11. *If* $f(x) \leq g(x)$ *for* $x \to a$,

$$\lim_{x \to a} f(x) = F \quad \textit{and} \quad \lim_{x \to a} g(x) = G,$$

then $F \leq G$.

This corollary also holds for $x \to a^{\pm}$ and $x \to \pm\infty$.

Proof. If $f(x) \leq g(x)$ for $x \to a$, then $f(x) - g(x) \leq 0$ for $x \to a$.
By the Laws of Limits

$$\lim_{x \to a} f(x) - g(x) = F - G$$

and therefore by Theorem 3.10 (page 68), $F - G \leq 0$. Q.E.D.

The next corollary is a particular case of Corollary 3.11.

Corollary 3.12. *a. If*

$$f(x) \leq Q \quad \textit{for } x \to a \quad \textit{and} \quad \lim_{x \to a} f(x) = F,$$

then $F \leq Q$.

b. If

$$Q \leq g(x) \quad \textit{for } x \to a \quad \textit{and} \quad \lim_{x \to a} g(x) = G,$$

then $Q \leq G$.

Both results also hold for $x \to a^{\pm}$ *and* $x \to \pm\infty$.

Proof. If $g(x) = Q$ is a constant function, then the former follows from Corollary 3.11.
If $f(x) = Q$ is a constant function, then the latter follows from Corollary 3.11. Q.E.D.

Observe that Corollary 3.11 holds only if the limits of f and g exist. It is incorrect to conclude that

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \quad \text{if} \quad f(x) \leq g(x) \quad \text{for } x \to a.$$

Indeed, we have that

$$\sin\left(\frac{1}{x}\right) \leq 1 \quad \text{for any nonzero } x.$$

However, by Example 1.9 (page 17)

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \neq K \quad \text{for every } K.$$

Since, we can only compare two numbers

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \not\leq 1.$$

The **Squeeze Theorem** stated below is similarly to Corollary 3.11 (page 69).

Theorem 3.13. *If* $\quad h(x) \leq f(x) \leq g(x) \quad$ *for* $x \to a$*, and*

$$\lim_{x\to a} h(x) = L = \lim_{x\to a} g(x),$$

then

$$\lim_{x\to a} f(x) = L.$$

This theorem also holds for $x \to a^{\pm}$ and $x \to \pm\infty$.

Proof. Let $\varepsilon > 0$ be any positive number. For this ε there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$|h(x) - L| < \varepsilon \quad \text{for} \quad 0 < |x - a| < \delta_1$$

and

$$|g(x) - L| < \varepsilon \quad \text{for} \quad 0 < |x - a| < \delta_2$$

If $\delta = \min(\delta_1, \delta_2) > 0$, both statements hold for $0 < |x - a| < \delta$, thus

$$L - \varepsilon < h(x) < L + \varepsilon \quad \text{and} \quad L - \varepsilon < g(x) < L + \varepsilon.$$

Hence, for $0 < |x - a| < \delta$

$$L - \varepsilon < h(x) < f(x) < g(x) < L + \varepsilon.$$

Therefore,

$$|f(x) - L| < \varepsilon \quad \text{for} \quad 0 < |x - a| < \delta,$$

and $\lim\limits_{x\to a} f(x) = L$. Q.E.D.

In the next example, we give the result we promised in Example 3.4 (page 60).

Example 3.12. Let $h(x) = x \sin\left(\frac{1}{x}\right)$.

Since, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for every nonzero x, we have

$$-x \leq x \sin\left(\frac{1}{x}\right) \leq x \quad \text{for every } x > 0$$

and

$$x \leq x \sin\left(\frac{1}{x}\right) \leq -x \quad \text{for every } x < 0.$$

Since $\lim\limits_{x \to 0} x = 0 = \lim\limits_{x \to 0} -x$ we conclude that

$$\lim_{x \to 0^+} x \sin\left(\frac{1}{x}\right) = 0 = \lim_{x \to 0^-} x \sin\left(\frac{1}{x}\right).$$

Therefore,

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

$\square$

Exercises III

1. Prove that if $f(x) = c$ for a constant c, then

$$\lim f(x) = c.$$

2. Prove Proposition 3.1 (page 46), for $x \to a^+, x \to a^-$ and $x \to -\infty$.

3. Prove

   a. part (a) of Theorem 3.2 (page 47), for $x \to a^+$.

   b. part (b) of Theorem 3.2, for $x \to a^-$.

   c. part (c) of Theorem 3.2, for $x \to a^-$.

d. part (d) of Theorem 3.2, for $x \to -\infty$.

e. part (e) of Theorem 3.2, for $x \to a^-$.

4. Prove that for any number a

$$\lim_{x \to a} x = a.$$

5. Prove that for any nonzero number a

$$\lim_{x \to a} \frac{1}{x} = \frac{1}{a}.$$

6. Prove that the converse of part (a) of Theorem 3.2 (page 47) does not hold. That is, give two functions $f(x)$, $g(x)$ and a number a such that the limit

$$\lim_{x \to a}[f(x) + g(x)] \quad \text{exist}$$

and either

$$\lim_{x \to a} f(x) \quad \text{or} \quad \lim_{x \to a} g(x) \quad \text{does not exist.}$$

7. Give the definitions of the expressions

a. $\lim_{x \to a^-} f(x) = R^+$
$f(x)$ tends to R from the right as x tends to a from the left.

b. $\lim_{x \to a^+} f(x) = L^-$
$f(x)$ tends to L from the left as x tends to a from the right.

c. $\lim_{x \to \infty} f(x) = L^-$
$f(x)$ tends to L from the left as x tends to infinity (increases without bound).

d. $\lim_{x \to -\infty} f(x) = L^-$
$f(x)$ tends to L from the left as x tends to negative infinity (decreases without bound).

8. Give the graphic representations of the expressions listed in Exercise 7.

9. Prove without using part (b) of Theorem 3.2 (page 47) that if

$$\lim_{x \to a} f(x) = K \quad \text{and } c \text{ is any constant,}$$

then

$$\lim_{x \to a}[cf(x)] = cK.$$

10. Give the negation of Definition 3.4 (page 56).

11. Prove part (a) of Theorem 3.7 (page 63), for

$$\lim_{x \to a^{\pm}} g(x) = b^{+} \quad \text{and} \quad \lim_{x \to -\infty} g(x) = b^{+}.$$

12. Prove that if $g(x) \to R^{+}$ as $x \to a$, then $g(x) \to R$ as $x \to a$.

13. Give a counterexample to show that the converse of the implication of Exercise 12 is false.

14. Prove Theorem 3.10 (page 68), for $x \to a^{\pm}$ and $x \to -\infty$.

15. Explain why if a number c is such that

$$0 < c < \delta \quad \text{for every } \delta > 0,$$

then $c = 0$.

Complete solutions are provided on page 286.

Chapter 4
Infinite Limits

Since only unbounded functions may have infinite limits, we provide for completeness, the definition of bounded functions.

Definition 4.1. Let D_f be the domain of a function f. The function f is

a. *bounded above* if there is a number B_U called *upper bound* such that

$$f(x) \leq B_U \qquad \text{for every} \quad x \in D_f.$$

b. *bounded below* if there is a number B_L called *lower bound* such that

$$f(x) \geq B_L \qquad \text{for every} \quad x \in D_f.$$

c. *bounded* if there is a number B called *bound* such that

$$-B \leq f(x) \leq B \qquad \text{for every} \quad x \in D_f.$$

A function may also be bounded on a subset of its domain.

Definition 4.2. Let S be a subset of the domain D_f of a function f.
The function f is

a. *bounded above on S* if there is a number B_U called *upper bound* such that

$$f(x) \leq B_U \qquad \text{for every} \quad x \in S.$$

b. *bounded below on S* if there is a number B_L called *lower bound* such that

$$f(x) \geq B_L \qquad \text{for every} \quad x \in S.$$

c. *bounded on S* if there is a number B called *bound* such that

$$-B \leq f(x) \leq B \qquad \text{for every} \quad x \in S.$$

Note. A function is bounded (above)(below) if it is bounded (above)(below) on its *domain*.

Example 4.1. The interval $[-1, 1]$ is the domain of the sine and cosine inverse functions. Since,

$$|\sin^{-1} x| \leq \frac{\pi}{2} \quad \text{for every} \quad -1 \leq x \leq 1.$$

The sine inverse function is bounded with bound $\pi/2$.
Since,

$$0 \leq \cos^{-1} x \leq \pi \quad \text{for every} \quad -1 \leq x \leq 1.$$

The cosine inverse function is bounded below and above with lower bound zero and upper bound π. Hence, it is bounded with bound π. See Exercise 1 (page 99) of this chapter. □

Example 4.2. The function

$$\sin\left(\frac{1}{x}\right) \qquad \text{of Example 1.9 (page 17)}$$

is bounded with bound $B = 1$. □

Example 4.3. The tangent function is bounded on the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, since

$$|\tan x| \leq 1 \quad \text{for every } -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}.$$

□

The negation of part (c) of Definition 4.1 yields the definition of an unbounded function.

Negation of Definition 4.1. (page 74)

Let D_f be the domain of a function f. The function f is

a. *unbounded above* if for any number H_U,

$$f(x) > H_U \quad \text{for some } x \in D_f.$$

b. *unbounded below* if for any number H_L,

$$f(x) < H_U \quad \text{for some } x \in D_f.$$

c. *unbounded* if for any number H, either

$$f(x) > H \;\text{ or }\; f(x) < H \quad \text{for some } x \in D_f.$$

The negation of part (c) of Definition 4.2 yields the definition of an unbounded function on a subset of its domain.

Negation of Definition 4.2. (page 74)

Let S be a subset of the domain D_f of a function f. The function f is

a. *unbounded above on* S if for any number H_U

$$f(x) > H_U \quad \text{for some } x \in S.$$

b. *unbounded below on* S if for any number H_L

$$f(x) < H_U \quad \text{for some } x \in S.$$

c. *unbounded on* S if for any number H,

$$f(x) > H \ \text{ or } \ f(x) < H \quad \text{for some } x \in S.$$

Remark 4.3. *1. To prove that a function f which is bounded below with bound B_L is unbounded above, we must show that for any number $V > B_L$*

$$f(x) > V \quad \textit{for some } x \in D_f.$$

Note that if $V \leq B_L$, then there is nothing to prove because in this case

$$V \leq B_L \leq f(x) \quad \textit{for some } x \in D_f.$$

2. Similarly, to prove that a function f which is bounded above with bound B_U is unbounded below, we must show that for any number $V < B_U$

$$f(x) < V \quad \textit{for some } x \in D_f.$$

Note that if $V \geq B_U$, then there is nothing to prove because in this case

$$V \geq B_L \geq f(x) \quad \textit{for every } x \in D_f.$$

Example 4.4. The domain of the function

$$f(x) = \sqrt{x} \quad \text{is the interval } [0, \infty).$$

The function is bounded below with lower bound 0, because

$$f(x) = \sqrt{x} \geq 0 \quad \text{for all } x \geq 0.$$

It is unbounded above, because for any number $M > 0$, there is $u = (M+1)^2 > 0$, such that

$$f(u) = \sqrt{(M+1)^2} = M + 1 > M.$$

□

Example 4.5. The tangent function is unbounded because for any number M there is

$$-\frac{\pi}{2} < x = \tan^{-1}(M+1) < \frac{\pi}{2},$$

such that

$$\tan x = \tan(\tan^{-1}(M+1)) = M + 1 > M.$$

□

Graphs of Bounded and Unbounded Functions

In the figures below, we present the graphs of some bounded and unbounded functions with domains $\mathbb{R}$.

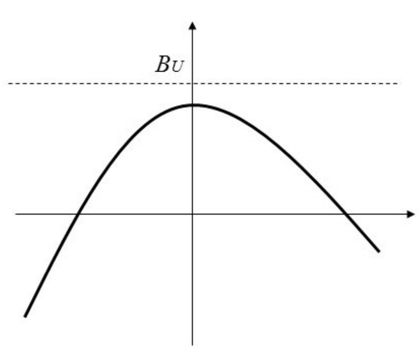


Figure 4.1: Graph of a bounded above and unbounded below function

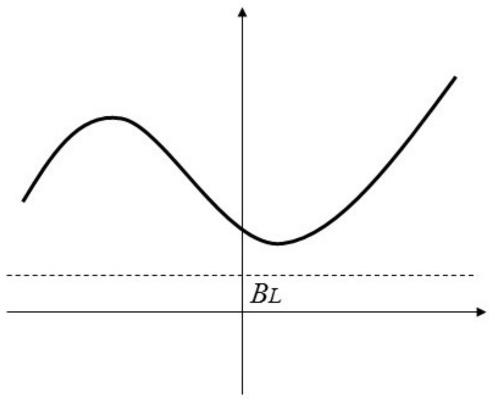


Figure 4.2: Graph of a bounded below and unbounded above function

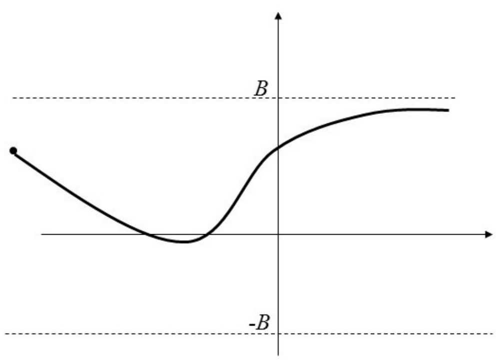


Figure 4.3: Graph of a bounded function

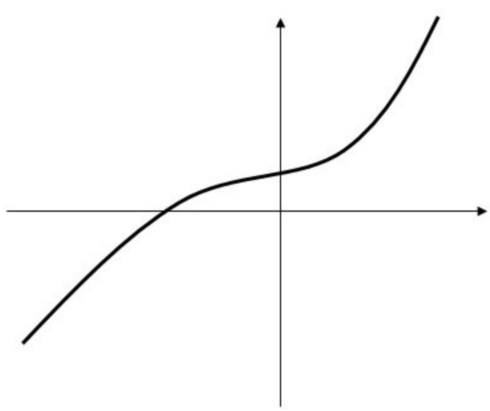


Figure 4.4: Graph of an unbounded function

Vertical Asymptotes

Infinite limits at a number are related to vertical asymptotes because for these limits the values of the function increase or decrease without bound.

We list four possible different cases for the vertical line $x = V$ to be a vertical asymptote.

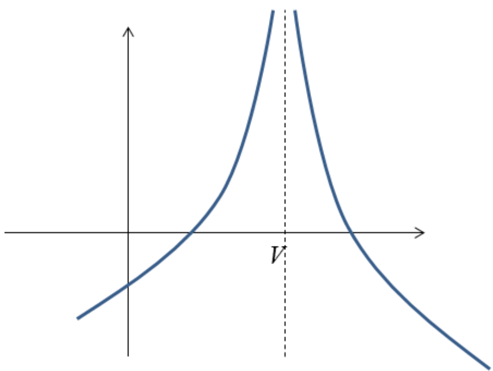


Figure 4.5: The function increases without bound around V

In Figure 4.5, the function, say f, increases without bound for numbers x on the right and left of the number V. Hence,

$$f(x) \to \infty \quad \text{for} \quad x \to V.$$

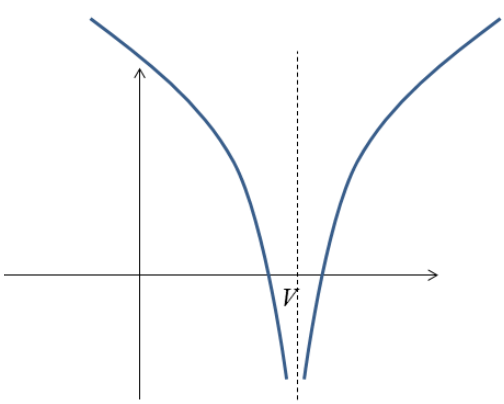


Figure 4.6: The function decreases without bound around V

In Figure 4.6, the function f, decreases without bound for numbers on the right and left of the number V. Hence,

$$f(x) \to -\infty \quad \text{for} \quad x \to V.$$

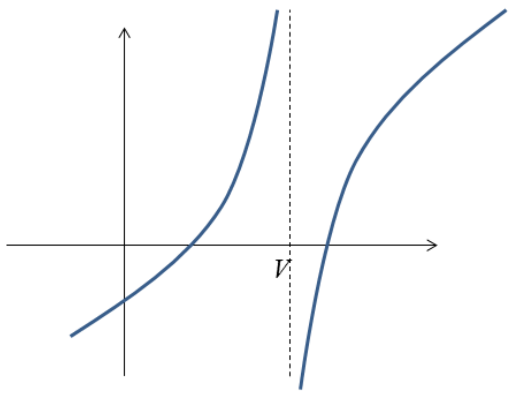


Figure 4.7: The function increases without bound on the left of V and decreases without bound on the right of V

In Figure 4.7, the function f, increases without bound for numbers on the left of the number V. Hence,

$$f(x) \to \infty \quad \text{for} \quad x \to V^-.$$

The function f, decreases without bound for numbers on the right of the number V. Hence,

$$f(x) \to -\infty \quad \text{for} \quad x \to V^+.$$

In Figure 4.8, the function f, decreases without bound for numbers on the left of the number V. Hence,

$$f(x) \to -\infty \quad \text{for} \quad x \to V^-.$$

The function f, increases without bound for numbers on the right of the number V. Hence,

$$f(x) \to \infty \quad \text{for} \quad x \to V^+.$$

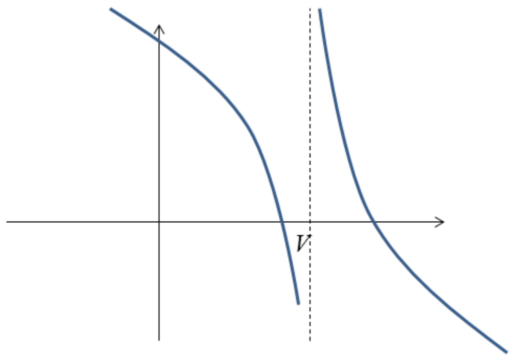


Figure 4.8: The function decreases without bound on the left of V and increases without bound on the right of V

It is clear from Figures 4.5 to 4.8 (pages 79 to 81) that a bounded function cannot have a vertical asymptote.

In the next definition we use, again, the infinity symbol (∞) to represent a function whose values increase/decrease "without bound."

Intuitively, the line $y = V$ is a *vertical asymptote* of the function $f(x)$ if

> we can make the values of $f(x)$ to increase or decrease without bound by taking numbers x close to the number V.

This statement is *not* the definition of a vertical asymptote. It merely expresses how we interpret a vertical asymptote.

The definitions of infinite limits at a number are given below.

Definition 4.4. a. The *limit of a function f at V from the right is infinite* if for any positive number $M > 0$ there is a positive number $\delta > 0$ such that

$$f(x) > M \quad \text{for every} \quad 0 < x - V < \delta. \tag{4.1}$$

We write,

$$\lim_{x \to V^+} f(x) = \infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to V^+.$$

Figure A.16 (page 270) shows the graphic representation of $\lim_{x \to V^+} f(x) = \infty$.

b. The *limit of a function f at V from the left is infinite* if for any positive number $M > 0$ there is positive number $\delta > 0$ such that

$$f(x) > M \quad \text{for every} \quad 0 < V - x < \delta. \tag{4.2}$$

We write,

$$\lim_{x \to V^-} f(x) = \infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to V^-.$$

Figure A.17 (page 270) shows the graphic representation of $\lim_{x \to V^-} f(x) = \infty$.

c. The *limit of a function f at V is infinite* if for any positive number $M > 0$ there is positive number $\delta > 0$ such that

$$f(x) > M \quad \text{for every} \quad 0 < |x - V| < \delta. \tag{4.3}$$

We write,

$$\lim_{x \to V} f(x) = \infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to V.$$

Figure A.18 (page 271) shows the graphic representation of $\lim_{x \to V} f(x) = \infty$.

We have the corresponding definitions for negative infinity.

Definition 4.5. a. The *limit of a function f at V from the right is negative infinite* if for any negative number $N < 0$ there is a positive number $\delta > 0$ such that

$$f(x) < N \quad \text{for every} \quad 0 < x - V < \delta. \tag{4.4}$$

We write,

$$\lim_{x \to V^+} f(x) = -\infty \qquad \text{or} \qquad f(x) \to -\infty \quad \text{as} \quad x \to V^+.$$

Figure A.19 (page 271) shows the graphic representation of $\lim_{x \to V^+} f(x) = -\infty$.

b. The *limit of a function f at V from the left is negative infinite* if for any negative number $N < 0$ there is a positive number $\delta > 0$ such that

$$f(x) < N \quad \text{for every} \quad 0 < V - x < \delta. \tag{4.5}$$

$$\lim_{x \to V^-} f(x) = -\infty \qquad \text{or} \qquad f(x) \to -\infty \quad \text{as} \quad x \to V^-.$$

Figure A.20 (page 272) shows the graphic representation of $\lim\limits_{x \to V^-} f(x) = -\infty$.

c. The *limit of a function f at V is negative infinite* if for any negative number $N < 0$ there is a positive number $\delta > 0$ such that

$$f(x) < N \quad \text{for every} \quad 0 < |x - V| < \delta. \tag{4.6}$$

$$\lim_{x \to V} f(x) = -\infty \qquad \text{or} \qquad f(x) \to -\infty \quad \text{as} \quad x \to V.$$

Figure A.21 (page 272) shows the graphic representation of $\lim\limits_{x \to V} f(x) = -\infty$.

Parts (a) of Definitions 4.4 and 4.5 only make sense, if the function f is defined for $x \to V^+$, parts (b) if it is defined for $x \to V^-$ and parts (c) if it is defined for $x \to V$.

Definition 4.6. A vertical line $x = V$ is a *vertical asymptote* of a function f if and only if any of the limits from (4.1) to (4.6) hold.

Refer to the graph of the function $f(x) = \dfrac{1}{x}$ for the next example.

Example 4.6. For any positive number $V > 0$, there is the positive number $\delta = \dfrac{1}{V}$ such that

$$\frac{1}{x} > V \quad \text{for every} \quad 0 < x < \delta = \frac{1}{V} > 0.$$

Hence, by part (a) of Definition 4.4

$$\lim_{x \to 0^+} \frac{1}{x} = \infty.$$

Similarly, for any negative number $U < 0$, there is the positive number $\delta = -\dfrac{1}{U} > 0$ such that

$$\frac{1}{x} < U \quad \text{for every} \quad -x < \delta = -\frac{1}{U}.$$

Hence, by part (b) of Definition 4.5

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

The line $x = 0$ is a vertical asymptote of the function $f(x) = \dfrac{1}{x}$. $\square$

It is not true that if a function is not defined at a number V, then the line $x = V$ may be a vertical asymptote.

In the next example we give a function defined on $\mathbb{R}$ with a vertical asymptote.

Example 4.7. Let

$$f(x) = \begin{cases} \frac{1}{x^2} & if \quad x \neq 0 \\ 0 & if \quad x = 0. \end{cases}$$

The domain of this function is $\mathbb{R}$ and $x = 0$ is a vertical asymptote because as we will see in Proposition 4.15 (page 97)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

□

The negations of Definitions 4.4 and 4.5 yields the definitions of non-infinite limits at a number. Below is the negation of part (c) of Definition 4.4, the negations of parts (a) and (b) together with the negation of Definition 4.5 are left as exercises. See Exercise 4 of this chapter.

Negation of part (c) of Definition 4.4 (page 81)

The limit

$$\lim_{x \to V} f(x) \neq \infty \quad \text{is not infinite}$$

if there is a positive number $M > 0$ such that for any positive number $\delta > 0$

$$f(x) \leq M \quad \text{for some} \quad 0 < |x - V| < \delta. \tag{4.7}$$

If a limit is non-infinite, then it may be either finite or non-existing; non necessarily finite.

We established in Chapters 1 and 2 that a limit does not exist if it is not equal to a number. In the next proposition we prove that if a limit is infinite, then it does not exist.

Proposition 4.7. *If* $\lim_{x \uparrow V} f(x) = \pm\infty$*, then* $\lim_{x \uparrow V} f(x) \neq K$ *for any* K*.*

Proof. We prove the contrapositive for $x \to V$. The remaining cases are similar.

We assume that $\lim_{x \to V} f(x) = K$ for some K and prove that $\lim_{x \to V} f(x) \neq \infty$.

For the positive number $\varepsilon = |K| + 1 > 0$ there is a positive number $\delta_1 > 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for every} \quad 0 < |x - V| < \delta_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$, we have for every $0 < |x - V| < \delta_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K \leq 2|K| + 1.$$

For any positive number $\delta > 0$, we have either $\delta < \delta_1$ or $\delta_1 \leq \delta$.

If $\delta < \delta_1$, there is a number x such that $|x - V| < \delta < \delta_1$.

If $\delta_1 \leq \delta$, there is a number x is such that $0 < |x - V| < \delta_1 \leq \delta$.

In either case, there is $M = (2|K| + 1) > 0$ such that

$$f(x) < M \quad \text{for some} \quad |x - V| < \delta.$$

Hence,

$$\lim_{x \to V} f(x) \neq \infty \quad \text{by (4.7).}$$

Q.E.D.

The converse of Proposition 4.7 is not true.

By Example 1.9 (page 17) and Example 4.2 (page 75), the limit

$$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist and it is not infinite.}$$

Students may not be able to explain why a limit does not exists. But they may be able to explain why a limit is infinite. Hence, we should be asking them to explain why a limit is infinite, or better, to prove it.

The next proposition establishes the relationship between infinite limits at a number and infinite limits at the right and left of a number.

Proposition 4.8. *a.* $\lim_{x \to V} f(x) = \infty$ *if and only if*

$$\lim_{x \to V^+} f(x) = \infty = \lim_{x \to V^-} f(x).$$

b. $\lim_{x \to V} f(x) = -\infty$ *if and only if*

$$\lim_{x \to V^+} f(x) = -\infty = \lim_{x \to V^-} f(x).$$

Proof. We provide the proof of part (a). The proof of part (b) is similar.

($\Rightarrow$) For any positive number $M > 0$ there is a positive number $\delta > 0$ such that

$$f(x) > M \quad \text{for every} \quad 0 < |x - V| < \delta.$$

Hence,

$$f(x) > M \quad \text{for every} \quad 0 < x - V < \delta$$

and

$$f(x) > M \quad \text{for every} \quad 0 < V - x < \delta.$$

By parts (a) and (b) of Definition 4.1 (page 74)

$$\lim_{x \to V^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to V^-} f(x) = \infty.$$

($\Leftarrow$) For any positive number $M > 0$, there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$f(x) > M \quad \text{for every} \quad 0 < x - a < \delta_1$$

and

$$f(x) > M \quad \text{for every} \quad 0 < a - x < \delta_2.$$

Both statements hold for $0 < |x - V| < \delta$ where $\delta = \min(\delta_1, \delta_2) > 0$. Thus,

$$f(x) > M \quad \text{for every} \quad 0 < |x - V| < \delta.$$

By part (c) of Definition 4.1 (page 74)

$$\lim_{x \to V} f(x) = \infty.$$

Q.E.D.

Negation of Proposition 4.8 (page 86).

$\lim_{x \to V} f(x) \neq \infty$ if and only if

a. $\lim_{x \to V^+} f(x) \neq \infty$, or

b. $\lim_{x \to V^-} f(x) \neq \infty$, or

c. $\lim_{x \to V^+} f(x)$ and $\lim_{x \to V^-} f(x)$ are infinite limits but not equal.

$\lim_{x \to V} f(x) \neq -\infty$ if and only if

a. $\lim_{x \to V^+} f(x) \neq -\infty$ or

b. $\lim_{x \to V^-} f(x) \neq -\infty$ or

c. $\lim_{x \to V^+} f(x)$ and $\lim_{x \to V^-} f(x)$ are infinite limits but not equal.

Students may find difficult to prove that a limit is not infinite, but they may be capable of proving or explaining parts (c) of the negation above.

Infinite Limits at Infinity

In this section we consider functions which increase/decrease without bound as the independent variable also increase/decrease without bound. Some particular cases are shown in the two figures below.

In Figure 4.9, the function increases without bound as x increases without bound, and the function decreases without bound as x decreases without bound. Hence,

$$f(x) \to \infty \quad \text{as} \quad x \to \infty \tag{4.8}$$

and

$$f(x) \to -\infty \quad \text{as} \quad x \to -\infty \tag{4.9}$$

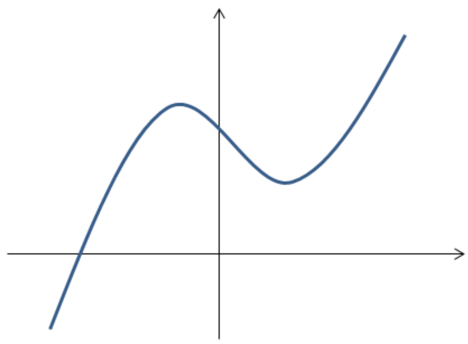


Figure 4.9: The function increases and decreases *without bound*

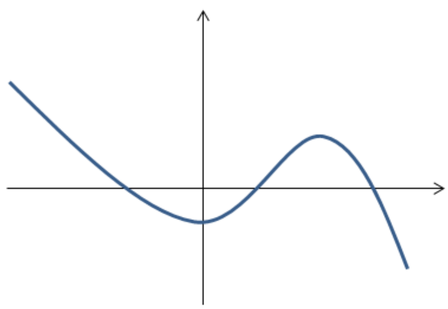


Figure 4.10: The function increases and decreases *without bound*

In Figure 4.10, the function decreases without bound as x increases without bound, and the function increases without bound as x decreases without bound. Hence,

$$f(x) \to -\infty \quad \text{as} \quad x \to \infty \tag{4.10}$$

and

$$f(x) \to \infty \quad \text{as} \quad x \to -\infty \tag{4.11}$$

Intuitively,

> we can make the values of the function $f(x)$ to increase or decrease by taking numbers x very large or very small (negatively).

Again, this statement merely expresses how we interpret the limits eqs. (4.8) to (4.11).

Their definitions are as follows.

Definition 4.9. a. The *limit of a function f at infinity is infinite* if for any positive number $M > 0$ there is a positive number $P > 0$ such that

$$f(x) > M \quad \text{for every} \quad x > P.$$

We write,

$$\lim_{x \to \infty} f(x) = \infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to \infty.$$

Figure A.22 (page 273) shows the graphic representation of $\lim_{x \to \infty} f(x) = \infty$.

b. The *limit of a function f at infinity is negative infinite* if for any negative number $N < 0$ there is a positive number $P > 0$ such that

$$f(x) < N \quad \text{for every} \quad x > P.$$

We write,

$$\lim_{x \to \infty} f(x) = -\infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to -\infty.$$

Figure A.23 (page 273) shows the graphic representation of $\lim_{x \to \infty} f(x) = -\infty$.

c. The *limit of a function f at negative infinity is infinite* if for any positive number $M > 0$ there is a negative number $Q < 0$ such that

$$f(x) > M \quad \text{for every} \quad x < Q.$$

We write,

$$\lim_{x \to -\infty} f(x) = \infty \qquad \text{or} \qquad f(x) \to \infty \quad \text{as} \quad x \to -\infty.$$

Figure A.24 (page 274) shows the graphic representation of $\lim_{x \to -\infty} f(x) = \infty$.

d. The *limit of a function f at negative infinity is negative infinite* if for any negative number $N < 0$ there is a positive number $Q < 0$ such that

$$f(x) < N \quad \text{for every} \quad x < Q.$$

We write,

$$\lim_{x \to -\infty} f(x) = -\infty \qquad \text{or} \qquad f(x) \to -\infty \quad \text{as} \quad x \to -\infty.$$

Figure A.25 (page 274) shows the graphic representation of $\lim_{x \to -\infty} f(x) = -\infty$.

Remark 4.10. *1. Parts (a) and (b) of Definition 4.9 only makes sense if the function f is defined on an infinite open interval (P, ∞) for some $P > 0$.*

2. Parts (c) and (d) only makes sense if the function is defined on an infinite open interval $(-\infty, Q)$ for some $Q < 0$.

Negation of Definition 4.9

a. $\lim_{x\to\infty} f(x) \neq \infty$
if there is a positive number $M > 0$ such that for any positive number $P > 0$

$$f(x) \leq M \quad \text{for some} \quad x > P.$$

b. $\lim_{x\to-\infty} f(x) \neq \infty$
if there is a positive number $M > 0$ such that for any negative number $Q < 0$

$$f(x) \leq M \quad \text{for some} \quad x < Q.$$

c. $\lim_{x\to\infty} f(x) \neq -\infty$
if there is a negative number $N < 0$ such that for any positive number $P > 0$

$$f(x) \geq N \quad \text{for some} \quad x > P.$$

d. $\lim_{x\to-\infty} f(x) \neq -\infty$
if there is a negative number $N < 0$ such that for any positive number $Q < 0$

$$f(x) \geq N \quad \text{for some} \quad x < Q.$$

In Chapter 2, we establish in (2.11) (page 35) that the limit $\lim_{x\to\infty} f(x)$ does not exist if

$$\lim_{x\to\infty} f(x) \neq K \quad \text{for any } K.$$

Similarly, the limit $\lim_{x\to-\infty} f(x)$ does not exist if

$$\lim_{x\to-\infty} f(x) \neq K \quad \text{for any } K.$$

Similarly to Proposition 4.7 (page 85) we have the following.

Proposition 4.11. *If* $\lim_{x\to\pm\infty} f(x) = \pm\infty$*, then* $\lim_{x\to\pm\infty} f(x) \neq K$ *for any* K.

Proof. We prove the contrapositive for positive infinite. The other cases are similar. We assume that

$$\lim_{x\to\infty} f(x) = K.$$

For the positive number $\varepsilon = |K| + 1 > 0$, there is a positive number $P_1 > 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for} \quad x > P_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$, we have that for any $x > P_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K \leq 2|K| + 1.$$

Let $P > 0$ be a positive number and $M = 2|K| + 1$.
If $P \geq P_1$, then there is a number $x > P \geq P_1$.
If $P < P_1$, then there is a number $x > P_1 > P$.
In either case,

$$f(x) \leq 2|K| + 1 = M \quad \text{for some} \quad x > P.$$

Hence, $\lim_{x\to\infty} f(x) \neq \infty$.

Now, we assume that

$$\lim_{x\to-\infty} f(x) = K.$$

For this positive number $\varepsilon = |K| + 1 > 0$, there is a negative number $Q_1 < 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for} \quad x < Q_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$, we have that for any $x < Q_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K \leq 2|K| + 1.$$

Let $Q < 0$ be any negative number and $M = 2|K| + 1$.
If $Q < Q_1$, then there is a number $x < Q < Q_1$.
If $Q \geq Q_1$, then there is a number $x < Q_1 \leq Q$.
In either case,

$$f(x) \leq 2|K| + 1 = M \quad \text{for some} \quad x < Q.$$

Hence, $\lim_{x\to-\infty} f(x) \neq \infty$. Q.E.D.

See that the limits

$$\lim_{x \to \pm\infty} \sin x \quad \text{do not exist and they are not infinite.}$$

Hence, the converse of Proposition 4.11 is not true.

If a limit is infinite, then it does not exist, but not the other way around.

> An infinite limit and a non-existing limit are two very different things.
> The limit $\lim f(x)$
>
> A. is infinite if it satisfies any one of the parts of Definition 4.4 (page 81), Definition 4.5 (page 82) or Definition 4.9 (page 89).
>
> B. does not exist if $\lim f(x) \neq K$ for any number K.

Reciprocal Functions

In this section we study the relationship between finite limits and reciprocal functions.

Definition 4.12. If $f(x)$ is a function with domain D_f, then its *reciprocal* is the function

$$F(x) = \frac{1}{f(x)} \quad \text{with domain} \quad D_F = \{x \in D_f | f(x) \neq 0\}.$$

If $\lim f(x) = 0$, then it not true that $\lim \dfrac{1}{f(x)}$ is infinite, as we show in the next two examples.

Example 4.8. Let δ_A be the following function.

$$\omega(x) = \begin{cases} x & if \quad x \in \mathbb{Q} \\ -x & if \quad x \in \mathbb{I}. \end{cases}$$

The domain of this function is $\mathbb{R}$. Hence, for any number a the limits

$$\lim_{x \uparrow a} \frac{1}{\omega(x)} \quad \text{are defined.}$$

By part (a) of Exercise 6 of this chapter,

$$\lim_{x \to 0} \omega(x) = 0.$$

Let $M = 1 > 0$.

For any positive number $\delta > 0$ there is an irrational x such that $0 < x < \delta$. Hence,

$$\Omega(x) = -\frac{1}{x} < 0 < M \quad \text{for some} \quad 0 < x < \delta.$$

By (4.7) on page 84, the limit

$$\lim_{x \to 0} \Omega(x) \neq \infty \quad \text{is not infinite.}$$

Similarly it can be proved that it is not negative infinite. See Exercise 12 of this chapter. □

Example 4.9. By Example 3.4 (page 60)

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

However, the limit

$$\lim_{x \to 0} \frac{1}{x \sin\left(\frac{1}{x}\right)} \qquad \text{is undefined.}$$

Indeed, we have that

$$\lim_{x \to \infty} \frac{1}{2\pi x} = 0.$$

Thus, for any positive number $\delta > 0$ there is a positive number $P > 0$ such that

$$\frac{1}{2\pi x} < \delta \quad \text{for} \quad x > P.$$

If $k > P$ is an integer, then

$$x = \frac{1}{2\pi k} < \delta \quad \text{and} \quad \sin\left(\frac{1}{x}\right) = \sin(2\pi k) = 0.$$

Therefore, for any positive number $\delta > 0$ there is a number $0 < x < \delta$ such that

$$\frac{1}{x \sin\left(\frac{1}{x}\right)} \qquad \text{is undefined}$$

and the function

$$\frac{1}{x\sin\left(\frac{1}{x}\right)} \qquad \text{is undefined for } x \to 0.$$

$\square$

In the next theorem we establish the conditions for the limit of a reciprocal function to be infinite.

Theorem 4.13. *a.* $\lim f(x) = 0^+$ *if and only if*

$$\lim \frac{1}{f(x)} = \infty.$$

b. $\lim f(x) = 0^-$ *if and only if*

$$\lim \frac{1}{f(x)} = -\infty.$$

Proof. We provide the proof of part (a) for $x \to V$ and $x \to \infty$. The proofs for $x \to V^\pm$ and $x \to -\infty$ are similar.

For $x \to V$.
($\Rightarrow$) Let $M > 0$ be any positive number. By part (c) of Definition 3.4 (page 56), for the positive number $\varepsilon = \frac{1}{M} > 0$ there is a positive number $\delta > 0$ such that

$$0 < f(x) < \varepsilon \qquad \text{for all} \quad 0 < |x - V| < \delta.$$

Hence,

$$\frac{1}{f(x)} > \frac{1}{\varepsilon} = M \qquad \text{for all} \quad 0 < |x - V| < \delta.$$

By part (c) of Definition 4.4 (page 81)

$$\lim_{x \to V} \frac{1}{f(x)} = \infty.$$

($\Leftarrow$) Let $\varepsilon > 0$ be any positive number. For the positive number $M = \frac{1}{\varepsilon} > 0$, there is a positive number $\delta > 0$ such that

$$\frac{1}{f(x)} > \frac{1}{\varepsilon} > 0 \quad \text{for} \quad 0 < |x - V| < \delta.$$

We then have that

$$f(x) > 0 \quad \text{and} \quad f(x) < \varepsilon \quad \text{for every } 0 < |x - V| < \delta.$$

Hence, by part (a) of Proposition 3.6 (page 61)

$$\lim_{x \to V} f(x) = 0^+.$$

For $x \to \infty$.
($\Rightarrow$) Let $M > 0$ by any positive number. By part (c) of Exercise 6 (page 71), for the positive number $\varepsilon = \dfrac{1}{M} > 0$ there is a positive number $P > 0$ such that

$$0 < f(x) < \varepsilon \qquad \text{for all} \quad x > P.$$

Hence,

$$\frac{1}{f(x)} > \frac{1}{\varepsilon} = M \qquad \text{for all} \quad x > P.$$

By part (c) of Definition 4.4 (page 81)

$$\lim_{x \to \infty} \frac{1}{f(x)} = \infty.$$

($\Leftarrow$) Let $\varepsilon > 0$ be any positive number. For the positive number $M = \dfrac{1}{\varepsilon} > 0$, there is a positive number $P > 0$ such that

$$\frac{1}{f(x)} > M = \frac{1}{\varepsilon} > 0 \quad \text{for} \quad x > P.$$

We then have that

$$f(x) > 0 \quad \text{and} \quad f(x) < \varepsilon \quad \text{for every } x > P.$$

Hence, by Proposition 3.6

$$\lim_{x \to \infty} f(x) = 0^+.$$

The proof of part (b) is similar. Q.E.D.

The conditions of Theorem 4.13 must be met to conclude. See that in Example 4.8, we stated that

$$\lim_{x \to 0} \omega(x) = 0.$$

and in Exercise 13 of this chapter, we ask you to prove that

$$\lim_{x\to 0} \frac{1}{\omega(x)} \neq \pm\infty.$$

Take note that Theorem 4.13 is not saying that

$$\lim f(x) = 0^+ \quad \Rightarrow \quad \lim \frac{1}{f(x)} = \frac{1}{0^+} = \infty.$$

It is very bad habit to divide by zero, regardless of the liberties we may wish to take with notation.

See how we apply Theorem 4.13 in the next example.

Example 4.10. By Example 3.3 (page 59)

$$\lim_{x\to 0^+} \sin x = 0^+ \quad \text{and} \quad \lim_{x\to 0^-} \sin x = 0^-.$$

By Theorem 4.13

$$\lim_{x\to 0^+} \frac{1}{\sin x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{\sin x} = -\infty.$$

Therefore, by the Negation of Proposition 4.8 (page 87)

$$\lim_{x\to 0} \csc x \quad \text{is neither } \infty \text{ nor } -\infty.$$

□

Remark 4.14. *Since the reciprocal of* $\frac{1}{f(x)}$ *is* $f(x)$, *Theorem 4.13 can be stated as follows.*

a. $\lim \frac{1}{f(x)} = 0^+$ *if and only if* $\lim f(x) = \infty$.

b. $\lim \frac{1}{f(x)} = 0^-$ *if and only if* $\lim f(x) = -\infty$.

The next proposition has important applications in the evaluation of limits of reciprocal functions.

Proposition 4.15. *We apply Theorem 4.13 to prove that*

$$\lim_{x\to 0} \frac{1}{x^n} = \infty \quad \textit{for any positive even integer } n. \tag{4.12}$$

$$\lim_{x\to 0^+} \frac{1}{x^n} = \infty \quad \textit{for any positive odd integer } n. \tag{4.13}$$

$$\lim_{x\to 0^-} \frac{1}{x^n} = -\infty \quad \textit{for any positive odd integer } n. \tag{4.14}$$

Proof. By Corollary 3.3 (page 53) for any positive integer n

$$\lim_{x\to 0} x^n = 0.$$

If n is even, then $x^n > 0$ for every $x \neq 0$ Thus, by Proposition 3.6 (page 61)

$$\lim_{x\to 0} x^n = 0^+.$$

By part (a) of Theorem 4.13

$$\lim_{x\to 0^+} \frac{1}{x^n} = \infty.$$

If n is odd, then $x^n > 0$ for $x > 0$ and $x^n < 0$ for $x < 0$. Thus, by Proposition 3.6

$$\lim_{x\to 0^+} x^n = 0^+ \quad \text{and} \quad \lim_{x\to 0^-} x^n = 0^-.$$

By Theorem 4.13

$$\lim_{x\to 0^+} \frac{1}{x^n} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x^n} = -\infty.$$

Q.E.D.

If $n = 2$, then as promised in Example 4.7 (page 84),

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

Students must be able to apply Theorem 4.13 (page 94), in order to sketch the graphs of functions and their reciprocals. They also must be made aware that

i. if $f(x) > 0$ for $x \to a^+$, then $\dfrac{1}{f(x)} > 0$ for $x \to a^+$.

ii. if $f(x) < 0$ for $x \to a^+$, then $\dfrac{1}{f(x)} < 0$ for $x \to a^+$.

Example 4.11. In Figure 4.11, we present the graphs of a function f and its reciprocal $\dfrac{1}{f(x)}$.

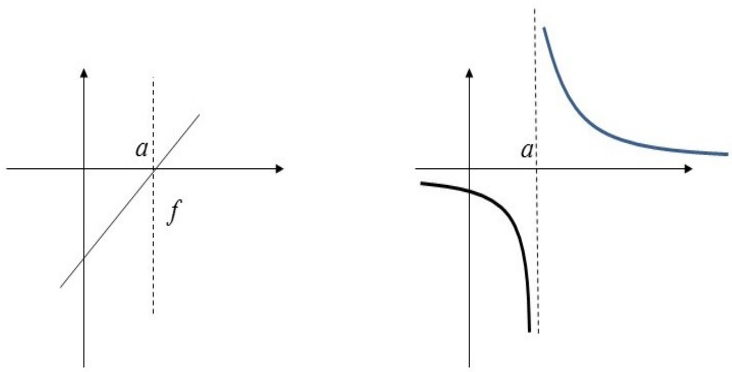


Figure 4.11: The graph of the function f and its reciprocal $\dfrac{1}{f(x)}$

- Since, $f(x) < 0$ for $x \to a^-$, its reciprocal $\dfrac{1}{f(x)}$ is negative for $x \to a^-$ and $\lim\limits_{x\to a^-} f(x) = -\infty$.

- Since, $f(x) > 0$ for $x \to a^+$, its reciprocal $\dfrac{1}{f(x)}$ is positive for $x \to a^+$ and $\lim\limits_{x\to a^-} f(x) = \infty$.

- Since, $f(x) < 0$ for $x \to -\infty$ and $\lim\limits_{x\to -\infty} f(x) = -\infty$, its reciprocal $\dfrac{1}{f(x)}$ is negative for $x \to -\infty$ and $\lim\limits_{x\to -\infty} \dfrac{1}{f(x)} = 0$.

- Since, $f(x) > 0$ for $x \to \infty$ and $\lim\limits_{x\to \infty} f(x) = \infty$, its reciprocal $\dfrac{1}{f(x)}$ is positive for $x \to \infty$ and $\lim\limits_{x\to \infty} \dfrac{1}{f(x)} = 0$. $\square$

Exercises

1. Prove that a function is bounded if and only if it is bounded above and below.
2. Give the negation of Definition 4.1.
3. Give the negations of parts (a) and (b) of Definition 4.4 (page 81).
4. Give the negation of Definition 4.5 (page 82).
5. Apply Definition 4.9 (page 89) to prove that

$$\lim_{x\to\infty} x = \infty \quad \text{and} \quad \lim_{x\to-\infty} x = -\infty.$$

6. Let $\omega(x)$ be the function defined in Example 4.8 (page 92).

   a. Prove that $\lim\limits_{x\to 0} \omega(x) = 0$.

   b. Prove that for any nonzero number a the limits $\lim\limits_{x\uparrow a} \omega(x)$ do not exist.

   c. Prove that $\lim\limits_{x\to\infty} \omega(x) \neq \pm\infty$.

   d. Prove that $\lim\limits_{x\to-\infty} \omega(x) \neq \pm\infty$.

7. Prove that for any function f

$$\lim f(x) = -\infty \quad \Rightarrow \quad \lim |f(x)| = \infty.$$

8. Prove that for any function f if

$$\lim_{x\uparrow V} f(x) = -\infty,$$

then the limits

$$\lim_{x\uparrow V} f(x) \quad \text{do not exist.}$$

That is, prove that if

$$\lim_{x\uparrow V} f(x) = -\infty,$$

then

$$\lim_{x\uparrow V} f(x) \neq K \quad \text{for any number } K.$$

9. Prove that for any function f if

$$\lim_{x \to -\infty} f(x) = \infty,$$

then the limit

$$\lim_{x \to -\infty} f(x) \quad \text{does not exist.}$$

That is, prove that if

$$\lim_{x \to -\infty} f(x) = \infty,$$

then

$$\lim_{x \to -\infty} f(x) \neq K \quad \text{for any number } K.$$

10. Prove part (a) of Theorem 4.13 (page 94) for $x \to V^{\pm}$.
11. Prove part (b) of Theorem 4.13 (page 94), for $x \to V$ and $x \to -\infty$.
12. Let $\omega(x)$ be the function defined in Example 4.8 (page 92).

    a. Apply the negation of part (c) of Definition 4.4 stated on page **??** to prove that

    $$\lim_{x \to 0} \Omega(x) = \lim_{x \to 0} \frac{1}{\omega(x)} \neq -\infty.$$

    b. Prove that

    $$\lim_{x \to 0} \Omega(x) = \lim_{x \to 0} \frac{1}{\omega(x)} \quad \text{does not exist.}$$

Complete solutions are provided on page 300.

Chapter 5
Properties of Infinite Limits

The incorrect applications of results about infinite limits arise from the use of the infinity symbol as if it represented a quantity.

In this chapter we present the correct application of the arithmetic operations of infinite limits.

Arithmetic Operations of Infinite Limits

The Laws of Limits (Theorem 3.2 (page 47) are for finite limits. We have similar results for infinite limits. Their applications are completely different.

Theorem 5.1. *If* $\lim a(x) = A$ *and* $\lim c(x) = C$ $(C \neq 0)$,

$\lim f(x) = \infty$, $\lim g(x) = \infty$,

$\lim u(x) = -\infty$, $\lim v(x) = -\infty$,

then

a. $\lim a(x) + f(x) = \infty$

b. $\lim a(x) + u(x) = -\infty$

c. $\lim f(x) + g(x) = \infty$

d. $\lim u(x) + v(x) = -\infty$

e. $\lim c(x)f(x) = \begin{cases} \infty & if \quad C > 0 \\ -\infty & if \quad C < 0. \end{cases}$

f. $\lim c(x)u(x) = \begin{cases} -\infty & if \quad C > 0 \\ \infty & if \quad C < 0. \end{cases}$

g. $\lim f(x)g(x) = \infty$

h. $\lim u(x)v(x) = \infty$

i. $\lim f(x)u(x) = -\infty$

Proof. We provide the proof for $x \to V$. The proofs for $x \to V^{\pm}$ and $x \to \pm\infty$ are similar.

a. Let $M > 0$ be any positive number. For this M, there is a positive number $\delta_1 > 0$ such that

$$|a(x) - A| < M \quad \text{for any } 0 < |x - V| < \delta_1.$$

Hence, by inequality 4 on page xii.

$$-M < a(x) - A < M \quad \text{for any } 0 < |x - V| < \delta_1.$$

By inequality 6 on page xii

$$-M - |A| \le -M + A < a(x) \quad \text{for any} \quad 0 < |x - V| < \delta_1. \tag{5.1}$$

For the positive number $M_1 = 2M + |A| > 0$, there is a positive number $\delta_2 > 0$, such that

$$f(x) > M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_2. \tag{5.2}$$

For $\delta = \min(\delta_1, \delta_2) > 0$, both inequalities (5.1) and (5.2) hold.

Thus,

$$a(x) + f(x) > -M - |A| + M_1 = M \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (a) follows from part (c) of Definition 4.4 (page 81).

b. Let $N < 0$ be any negative number. For the positive number $\varepsilon = -N > 0$, there is a positive number $\delta_1 > 0$ such that

$$|a(x) - A| < -N \quad \text{for any } 0 < |x - V| < \delta_1.$$

Hence, by inequality 4 on page xii

$$N < a(x) - A < -N \quad \text{for any } 0 < |x - V| < \delta_1$$

and by inequality 6 on page xii

$$a(x) < -N + A < -N + |A| \quad \text{for any} \quad 0 < |x - V| < \delta_1. \tag{5.3}$$

Since $2N < 0 < |A|$, the number $N_1 = 2N - |A| < 0$ is negative, and for this negative number, there is a positive number $\delta_2 > 0$, such that

$$u(x) < N_1 \quad \text{for any} \quad 0 < |x - V| < \delta_2. \tag{5.4}$$

For $\delta = \min(\delta_1, \delta_2) > 0$ both inequalities (5.3) and (5.4) hold.

Thus,

$$a(x) + u(x) < -N + |C| + N_1 = N \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (b) follows from part (c) of Definition 4.5 (page 82).

c. Let $M > 0$ be any positive number. For the positive number $M_1 = \dfrac{M}{2} > 0$, there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$f(x) > M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1$$

and

$$g(x) > M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_2.$$

For $\delta = \min(\delta_1, \delta_2) > 0$ both statements hold. Thus,

$$f(x) + g(x) > M_1 + M_1 = M \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (c) follows from part (c) of Definition 4.4 (page 81).

d. Let $N < 0$ be any negative number. For the negative number $N_1 = \dfrac{N}{2} < 0$, there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$u(x) < N_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1$$

and

$$v(x) < N_1 \quad \text{for any} \quad 0 < |x - V| < \delta_2$$

For $\delta = \min(\delta_1, \delta_2) > 0$ both statements hold. Thus,

$$u(x) + v(x) < N_1 + N_1 = N \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (d) follows from part (c) of Definition 4.5 (page 82).

e. Let $M > 0$ be any positive number. If $C > 0$, then for the positive number $M_1 = \dfrac{2M}{C} > 0$, there is a positive number $\delta_1 > 0$, such that

$$f(x) > M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1. \tag{5.5}$$

For the positive number $\varepsilon = \dfrac{C}{2} > 0$, there is a positive number $\delta_2 > 0$ such that

$$|c(x) - C| < \varepsilon \quad \text{for any} \quad 0 < |x - V| < \delta_2.$$

By inequality 4 on page on page xii

$$-\varepsilon < c(x) - C < \varepsilon \quad \text{for any} \quad 0 < |x - V| < \delta_2. \tag{5.6}$$

For $\delta = \min(\delta_1, \delta_2) > 0$ both statements (5.5) and (5.6) hold. Thus,

$$f(x)(c(x) - C) > -\varepsilon f(x) \quad \text{for any} \quad 0 < |x - V| < \delta$$

and

$$\begin{aligned} c(x)f(x) &= f(x)c(x) - Cf(x) + Cf(x) = f(x)(c(x) - C) + Cf(x) \\ &> -\varepsilon f(x) + Cf(x) = f(x)(C - \varepsilon) \\ &= f(x)\left(\frac{C}{2}\right) > M_1\left(\frac{C}{2}\right) \\ &= \frac{2M}{C}\left(\frac{C}{2}\right) = M \end{aligned}$$

Thus part (e) for $C > 0$ follows from part (c) of Definition 4.4 (page 81).

Let $N < 0$ be any negative number. If $C < 0$, then for the positive number $M_1 = \dfrac{2N}{C} > 0$, there is a positive number $\delta_1 > 0$, such that

$$f(x) > M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1. \tag{5.7}$$

For the positive number $\varepsilon = -\dfrac{C}{2} > 0$, there is $\delta_2 > 0$ such that

$$|c(x) - C| < \varepsilon \quad \text{for any} \quad 0 < |x - V| < \delta_2.$$

By inequality 4 on page xii

$$-\varepsilon < c(x) - C < \varepsilon \quad \text{for any} \quad 0 < |x - V| < \delta_2. \tag{5.8}$$

For $\delta = \min(\delta_1, \delta_2) > 0$ both statements (5.7) and (5.8) hold.

Thus,

$$f(x)(c(x) - C) < \varepsilon f(x) \quad \text{for any } 0 < |x - V| < \delta$$

and

$$\begin{aligned} c(x)f(x) &= f(x)c(x) - Cf(x) + Cf(x) = f(x)(c(x) - C) + Cf(x) \\ &< \varepsilon f(x) + Cf(x) = f(x)(C + \varepsilon) \\ &= f(x)\left(\frac{C}{2}\right) < M_1\left(\frac{C}{2}\right) \\ &= \frac{2N}{C}\left(\frac{C}{2}\right) = N \end{aligned}$$

Thus part (e) for $C < 0$ follows from part (c) of Definition 4.5 (page 82).

f. The proof of this part is left as an Exercise 5.

g. Let $M > 0$ be any positive number. For the positive number $M_1 = M_2 = \sqrt{M} > 0$, there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$f(x) > M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1$$

and

$$g(x) > M_2 \quad \text{for any} \quad 0 < |x - V| < \delta_2.$$

Hence, for $\delta = \min(\delta_1, \delta_2) > 0$ both statements hold and

$$f(x)g(x) > M_1M_2 = M \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (g) follows from part (c) of Definition 4.4 (page 81).

h. Let $M > 0$ be any positive number. For the negative number $M_1 = -\sqrt{M} < 0$, there are positive numbers $\delta_1, \delta_2 > 0$ such that

$$u(x) < M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1$$

and

$$v(x) < M_1 \quad \text{for any} \quad 0 < |x - V| < \delta_2.$$

Hence, for $\delta = \min(\delta_1, \delta_2) > 0$ both statements hold and

$$u(x)v(x) > M_1M_1 = M \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (h) follows from part (c) of Definition 4.4 (page 81).

i. Let $N < 0$ be any negative. For the negative number $N_1 = -\sqrt{-N} < 0$ there is a positive number $\delta_1 > 0$ such that

$$u(x) < N_1 \quad \text{for any} \quad 0 < |x - V| < \delta_1.$$

For the positive number $M = \sqrt{-N} > 0$, there is a positive number $\delta_2 > 0$ such that

$$f(x) > M \quad \text{for any} \quad 0 < |x - V| < \delta_2.$$

Hence, for $\delta = \min(\delta_1, \delta_2) > 0$ both statements hold and

$$u(x)f(x) < f(x)N_1 < MN_1 = N \quad \text{for any} \quad 0 < |x - V| < \delta.$$

Thus part (i) follows from part (c) of Definition 4.5 (page 82).

Q.E.D.

For the particular case where the functions $a(x)$ and $c(x)$ are constants, we have the following corollary.

Corollary 5.2. *If* $\lim f(x) = \infty$ *and* $\lim u(x) = -\infty$*, then*

a. $\lim Pf(x) = \infty$ *for any constant* $P > 0$.

b. $\lim Qf(x) = -\infty$ *for any constant* $Q < 0$.

c. $\lim Pu(x) = -\infty$ *for any constant* $P > 0$.

d. $\lim Qu(x) = \infty$ *for any constant* $Q < 0$.

e. $\lim f(x) + C = \infty$ *for any constant* C.

f. $\lim u(x) + C = -\infty$ *for any constant* C.

Students must know the correct applications of Theorem 5.1 and Corollary 5.2.

Theorem 5.1 is *not* saying that if $\lim f(x) = \infty$ and $\lim g(x) = \infty$, then

$$\lim f(x) + g(x) = \lim f(x) + \lim g(x) = \infty + \infty \quad \text{this is incorrect.}$$

Any similar applications of Theorem 5.1 are incorrect. Students should never applied this theorem in this way.

Example 5.1. By Exercise 4 on page 99, we have that

$$\lim_{x\to\infty} x = \infty.$$

Hence, by part (g) of Theorem 5.1,

$$\lim_{x\to\infty} x^n = \infty \quad \text{for any positive integer } n.$$

Similarly, by the same exercise

$$\lim_{x\to-\infty} x = -\infty.$$

Thus, by parts (h) and (i) of Theorem 5.1,

$$\lim_{x\to-\infty} x^n = \infty \quad \text{for any even positive integer } n.$$

and

$$\lim_{x\to-\infty} x^n = -\infty \quad \text{for any odd positive integer } n.$$

□

We apply parts (g), (h) and (i) of Theorem 5.1 to the product

$$f(x)\left(\frac{1}{g(x)}\right) = \frac{f(x)}{g(x)}$$

to evaluate limits of the quotient of two functions.

Example 5.2. By Example 4.10 (page 96)

$$\lim_{x\to0^+} \csc x = \lim_{x\to0^+} \frac{1}{\sin x} = \infty$$

and

$$\lim_{x\to0^+} \cos^{-1} x = \frac{\pi}{2} > 0.$$

Hence, by part (e) of Theorem 5.1

$$\lim_{x\to0^+} \frac{\cos^{-1} x}{\sin x} = (\lim_{x\to0^+} \cos^{-1} x)(\lim_{x\to0^+} \csc x) = \infty.$$

It is incorrect to apply Theorem 5.1 in the following manner.

$$\lim_{x\to0^+} \frac{\cos^{-1} x}{\sin x} = (\lim_{x\to0^+} \cos^{-1} x)(\lim_{x\to0^+} \csc x) = \left(\frac{\pi}{2}\right)\infty = \infty.$$

The product of a number times the infinity symbol does not make sense. □

We will show in the next three examples that if

$$\lim_{x \uparrow V} f(x) = 0 \quad \text{and} \quad \lim_{x \uparrow V} g(x) = \pm\infty,$$

then the limit

$$\lim_{x \uparrow V} f(x)g(x)$$

may be finite, infinite or does not exist.

Example 5.3. By Proposition 4.15 (page 97)

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0} \sin x = 0.$$

It is known that

$$\lim_{x \to 0^+} (\sin x) \left(\frac{1}{x} \right) = \lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

$\square$

Example 5.4. By Proposition 4.15 (page 97)

$$\lim_{x \to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0} \sin x = 0.$$

Then,

$$\frac{\sin x}{x^2} = \sin x \left(\frac{1}{x^2} \right).$$

We evaluate the limit of this function at zero from the right. As before

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

By part (a) of Theorem 5.1

$$\lim_{x \to 0^+} \frac{\sin x}{x^2} = \lim_{x \to 0^+} \frac{\sin x}{x} \left(\frac{1}{x} \right) = \infty.$$

$\square$

Example 5.5. By Example 3.4 (page 60), we have that $\displaystyle\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0.$

Hence, by the Laws of Limits

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)\right) = 0.$$

By Proposition 4.15 (page 97)

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

The product of these two functions is

$$x^2 \sin\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right) = \sin\left(\frac{1}{x}\right) = \sin\left(\frac{1}{x}\right)$$

and by Example 1.9 (page 17) the limit at zero of this product does not exist. □

We must be clear that if $\displaystyle\lim_{x\uparrow V} f(x) = \infty$ and $\displaystyle\lim_{x\uparrow V} g(x) = -\infty$, then the limit

$$\lim_{x\uparrow V} f(x) + g(x)$$

may be finite, infinite or does not exist, as we show in the next two examples.

Example 5.6. By Proposition 4.15 (page 97)

$$\lim_{x\to 0^-} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

We have that

$$\frac{1}{x^2} + \frac{1}{x} = \frac{1-x}{x^2} = (1-x)\left(\frac{1}{x^2}\right).$$

We apply Theorem 5.1 (page 101) with the limits

$$\lim_{x\to 0^-} (1-x) = 1 > 0 \quad \textit{and} \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

Thus,

$$\lim_{x\to 0^-} \frac{1}{x^2} + \frac{1}{x} = \lim_{x\to 0^-} (1-x)\left(\frac{1}{x^2}\right) = \infty.$$

□

Example 5.7. Let $f(x) = \dfrac{\sin x}{x^2}$ and $g(x) = \dfrac{1}{x}$. We have that

$$\lim_{x \to 0^+} -g(x) = -\infty$$

and

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

Hence, by part (f) of Theorem 5.1 (page 101)

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \left(\frac{\sin x}{x}\right)\left(\frac{1}{x}\right) = \infty.$$

By applying L'Hôpital's rule twice (covered in Chapter 6)

$$\begin{aligned} \lim_{x \to 0^+} f(x) - g(x) &= \lim_{x \to 0^+} \frac{\sin x - x}{x^2} \\ &= \lim_{x \to 0^+} \frac{\cos x - 1}{2x} = \lim_{x \to 0^+} \frac{-\sin x}{2} = 0. \end{aligned}$$

□

The next example is fundamental in the evaluation of polynomial and rational functions.

Example 5.8. By Example 5.1 and Remark 4.14

a. $\displaystyle\lim_{x \to \infty} \frac{1}{x^n} = 0^+$ for any integer n.

b. $\displaystyle\lim_{x \to -\infty} \frac{1}{x^n} = 0^+$ for any even positive integer n.

c. $\displaystyle\lim_{x \to -\infty} \frac{1}{x^n} = 0^-$ for any odd positive integer n.

□

Example 5.9. Let $P(x)$ be any polynomial of degree n, thus,

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + ax + a_0.$$

We factor x^n and obtain

$$P(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n}\right).$$

By Theorem 3.2 and Example 5.9

$$\lim_{x\to\pm\infty} \frac{b}{x^n} = 0 \quad \text{for any number } b.$$

Hence, by Theorem 3.2

$$\lim_{x\to\infty} \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n} \right) = a_n.$$

Since

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

By Example 5.9 and Theorem 5.1

a. $\lim_{x\to\infty} P(x) = \begin{cases} \infty & if \quad a_n > 0 \\ -\infty & if \quad a_n < 0. \end{cases}$ for any integer n.

b. $\lim_{x\to-\infty} P(x) = \begin{cases} \infty & if \quad a_n > 0 \\ -\infty & if \quad a_n < 0. \end{cases}$ for any even positive integer n.

c. $\lim_{x\to-\infty} P(x) = \begin{cases} \infty & if \quad a_n < 0 \\ -\infty & if \quad a_n > 0. \end{cases}$ for any odd positive integer n.

□

We use this example to explain what we promised in Example 1.9 (page 17).

Example 5.10. The function $g(x) = 4\pi x + \pi$ is linear with positive slope 4. By part (c) of Example 5.9,

$$\lim_{x\to\infty} 4\pi x + \pi = \infty.$$

By Theorem 4.13 (page 94)

$$\lim_{x\to\infty} \frac{1}{4\pi x + \pi} = 0^+.$$

Thus, by part (b) of the Laws of Limits (Theorem 3.2 on page 47)

$$\lim_{x\to\infty} \frac{2}{4\pi x + \pi} = 0.$$

By Definition 2.5 (page 35), for any $\delta > 0$, there is $M > 0$ such that

$$\left|\frac{2}{4\pi x + \pi}\right| < \delta \quad \text{for any} \quad x > M.$$

Similarly if $g(x) = 6\pi x + \pi$, then

$$\lim_{x\to\infty} \frac{2}{6\pi x + \pi} = 0.$$

Again, for any $\delta > 0$, there is $N > 0$ such that

$$\left|\frac{2}{6\pi x + \pi}\right| < \delta \quad \text{for any} \quad x > N.$$

For negative infinity, by part (c) of Example 5.9,

$$\lim_{x\to-\infty} 4\pi x + \pi = -\infty \quad \text{and} \quad \lim_{x\to-\infty} 6\pi x + \pi = -\infty.$$

Thus, by Theorem 4.13 (page 94),

$$\lim_{x\to-\infty} \frac{1}{4\pi x + \pi} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{6\pi x + \pi} = 0.$$

Hence, by Definition 2.5 (page 35), for any $\delta > 0$, there is $M, N < 0$ such that

$$\left|\frac{2}{4\pi x + \pi}\right| < \delta \quad \text{for any} \quad x < M.$$

and

$$\left|\frac{2}{6\pi x + \pi}\right| < \delta \quad \text{for any} \quad x < N.$$

$\square$

Inequalities

Theorem 5.3. *a. If $f(x) \leq g(x)$ for $x \to V$ and $\lim_{x\to V} f(x) = \infty$, then*

$$\lim_{x\to V} g(x) = \infty.$$

b. If $f(x) \leq g(x)$ for $x \to V$ and $\lim_{x\to V} g(x) = -\infty$, then

$$\lim_{x\to V} f(x) = -\infty.$$

c. *If* $f(x) \leq g(x)$ *for* $x \to \pm\infty$ *and* $\lim f(x) = \infty$*, then*

$$\lim_{x\to\pm\infty} g(x) = \infty.$$

d. *If* $f(x) \leq g(x)$ *for* $x \to \pm\infty$ *and* $\lim f(x) = -\infty$*, then*

$$\lim_{x\to\pm\infty} g(x) = -\infty.$$

This theorem also holds for $x \to V^{\pm}$.

Proof. We present the proofs of parts (a) and (b) for $x \to V$, the proofs for $x \to V^{\pm}$ and parts (c) and (d) are similar.

a. For any positive number $M > 0$, there is a positive number $\delta_1 > 0$ such that

$$f(x) > M \quad \text{for} \quad |x - V| < \delta_1.$$

Since $f(x) \leq g(x)$ for $x \to V$, there is a positive number $\delta_2 > 0$, such that

$$g(x) \geq f(x) \quad \text{for} \quad |x - V| < \delta_2.$$

If $\delta = \min(\delta_1, \delta_2) > 0$, then both statements hold for this $\delta > 0$, and

$$g(x) \geq f(x) > M \quad \text{for} \quad |x - V| < \delta.$$

Therefore

$$\lim_{x\to V} g(x) = \infty.$$

b. For any negative number $N < 0$, there is a positive number $\delta_1 > 0$ such that

$$g(x) < N \quad \text{for} \quad |x - V| < \delta_1.$$

Since $f(x) \leq g(x)$ for $x \to V$, there is a positive number $\delta_2 > 0$, such that

$$g(x) \geq f(x) \quad \text{for} \quad |x - V| < \delta_2.$$

If $\delta = \min(\delta_1, \delta_2) > 0$, then both statements hold for this $\delta > 0$, and

$$f(x) \leq g(x) < N \quad \text{for} \quad |x - V| < \delta.$$

Therefore

$$\lim_{x\to V} f(x) = -\infty.$$

Q.E.D.

Example 5.11. To evaluate the limit

$$\lim_{x \to 1/\pi^+} \frac{\csc\left(\frac{1}{x}\right)}{x - \frac{1}{\pi}}$$

we have that if $x \to 1/\pi^+$, then $1/x \to \pi^-$ and for this x,

$$\csc\left(\frac{1}{x}\right) \geq 1.$$

Hence, for $x \to 1/\pi^+$

$$\frac{\csc\left(\frac{1}{x}\right)}{x - 1/\pi} \geq \frac{1}{x - 1/\pi}.$$

Since

$$\lim_{x \to 1/\pi^+} \frac{1}{x - 1/\pi} = \infty$$

by part (a) of Theorem 5.3 (page 112)

$$\lim_{x \to 1/\pi^+} \frac{\csc\left(\frac{1}{x}\right)}{x - \frac{1}{\pi}} = \infty.$$

$\square$

Composition of Functions

For infinite limits, we have a theorem similar to Theorem 3.7 (page 63).

Theorem 5.4. *a. If* $\lim g(x) = b^+$ *and* $\lim_{y \to b^+} f(y) = \pm\infty$*, then*

$$\lim f(g(x)) = \pm\infty.$$

b. If $\lim g(x) = b^-$ *and* $\lim_{y \to b^-} f(y) = \pm\infty$*, then*

$$\lim f(g(x)) = \pm\infty.$$

c. *If* $\lim g(x) = b$ *and* $\lim_{y \to b} f(y) = \pm\infty$, *then*

$$\lim f(g(x)) = \pm\infty.$$

d. *If* $\lim g(x) = \infty$ *and* $\lim_{y \to \infty} f(y) = b$, *then*

$$\lim f(g(x)) = b.$$

e. *If* $\lim g(x) = -\infty$ *and* $\lim_{y \to -\infty} f(y) = b$, *then*

$$\lim f(g(x)) = b.$$

f. *If* $\lim g(x) = \pm\infty$ *and* $\lim_{y \to \pm\infty} f(y) = \pm\infty$, *then*

$$\lim f(g(x)) = \pm\infty.$$

Proof. We present the proofs of parts (a) and (b) for positive infinity and $x \to V$, the other proofs for $x \to V^{\pm}$ and $x \to \pm\infty$ are similar.

a. Let $M > 0$ be any positive number. There is a positive number $\delta_1 > 0$, such that

$$f(y) > M \quad \text{for} \quad 0 < y - b < \delta_1.$$

For this positive number $\delta_1 > 0$, there is a positive number $\delta > 0$, such that

$$0 < y = g(x) - b < \delta_1 \text{for} \quad |x - V| < \delta.$$

Hence,

$$f(g(x)) > M \quad \text{for} \quad |x - V| < \delta.$$

b. Let $M > 0$ be any positive number. There is a positive number $\delta_1 > 0$, such that

$$f(y) > M \quad \text{for} \quad 0 < b - y < \delta_1.$$

For this positive number $\delta_1 > 0$, there is a positive number $\delta > 0$, such that

$$0 < b - y = g(x) < \delta_1 \quad \text{for} \quad |x - V| < \delta.$$

Hence,

$$f(g(x)) > M \quad \text{for} \quad |x - V| < \delta.$$

The proof of part (c) is similar to parts (a) and (b).

We present the proof of parts (d) and (e) for $x \to V$ proofs for $x \to V^{\pm}$ and $x \to \pm\infty$ are similar.

d. Let $\varepsilon > 0$ be any positive number. There exists a positive number $P > 0$ such that

$$|f(y) - b| < \varepsilon \quad \text{for all} \quad y > P.$$

For any positive number $P > 0$ there is a positive number $\delta > 0$, such that

$$y = g(x) > P \quad \text{for all} \quad |x - V| < \delta.$$

Hence,

$$|f(g(x)) - b| < \varepsilon \quad \text{for all} \quad |x - V| < \delta$$

and

$$\lim_{x \to V} f(g(x)) = b.$$

e. Let $\varepsilon > 0$ be any positive number. There exists a positive number $Q < 0$ such that

$$|f(y) - b| < \varepsilon \quad \text{for all} \quad y < Q.$$

For any negative number $Q < 0$ there is a positive number $\delta > 0$, such that

$$y = g(x) < Q \quad \text{for all} \quad |x - V| < \delta.$$

Hence,

$$|f(g(x)) - b| < \varepsilon \quad \text{for all} \quad |x - V| < \delta$$

and

$$\lim_{x \to V} f(g(x)) = b.$$

Finally, we present the proof of part (f) for $x \to \infty$ and $\lim g(x) = \infty$ and $\lim\limits_{y \to \pm\infty} f(y) = \infty$.
The proofs for $x \to V$, $x \to V^{\pm}$ and $x \to -\infty$ and $\lim g(x) = -\infty$ and $\lim\limits_{y \to \pm\infty} f(y) = -\infty$.

f. Let $M > 0$ be any positive number. There is a positive number $P_1 > 0$ such that

$$f(y) > M \quad \text{for all} \quad y > P_1.$$

For this positive number $P_1 > 0$, there is a positive number $P > 0$, such that

$$y = g(x) > P_1 \quad \text{for all} \quad x > P.$$

Hence,

$$f(g(x)) > M \quad \text{for all} \quad x > P$$

and

$$\lim_{x \to \infty} f(g(x)) = \infty.$$

Q.E.D.

It is important to apply this theorem correctly. For example, in part (a), if we set $y = g(x)$, then $g(x) \to b^+$ as $x \to a$. Hence,

$$\lim_{x \to a} f(g(x)) = \lim_{y \to b^+} f(y) = \pm\infty.$$

The same for the other parts of this theorem.

It is incorrect to evaluate a function at infinity because ∞ does not represent a real number. Hence, in part (d)

$$\lim f(g(x)) = \lim f(\infty) = b \qquad \text{is incorrect,}$$

and

$$\lim f(g(x)) = f(\lim g(x)) = f(\infty) = b \quad \text{is even worse.}$$

Example 5.12. To evaluate the limit

$$\lim_{x \to 1/\pi^+} \csc\left(\frac{1}{x}\right)$$

We have that the function $\csc\left(\frac{1}{x}\right)$ is the composition of the functions $1/x$ and cosecant. In order to apply Theorem 5.4 (page 114), we consider the limit

$$\lim_{x \to 1/\pi^+} \frac{1}{x} = \pi.$$

If $x > \dfrac{1}{\pi}$, then $\dfrac{1}{x} < \pi$. Thus,

$$\lim_{x \to 1/\pi^+} \frac{1}{x} = \pi^-.$$

On the other hand

$$\lim_{y \to \pi^-} \csc(y) = \infty.$$

If $y = \dfrac{1}{x}$, then by part (a) of Theorem 5.4 (page 114)

$$\lim_{x \to 1/\pi^+} \csc\left(\frac{1}{x}\right) = \lim_{y \to \pi^-} \csc y = \infty.$$

$\square$

Exercises

1. Prove part (f) of Theorem 5.1 (page 101) for $x \to V$.
   Note. Assume parts (a) - (e) hold for $x \to V^{\pm}$ and $x \to \pm\infty$.

2. Prove part (g) of Theorem 5.1 (page 101) for $x \to \infty$.
   Note. Assume parts (a) - (f) hold for $x \to V^{\pm}$ and $x \to \pm\infty$.

3. Prove part (h) of Theorem 5.1 (page 101) for $x \to -\infty$.
   Note. Assume parts (a) - (g) hold for $x \to V^{\pm}$ and $x \to \pm\infty$.

4. a. Prove part (a) of Theorem 5.3 (page 112), for $x \to V^+$.

   b. Prove part (b) of Theorem 5.3 (page 112), for $x \to V^-$.

   c. Prove part (c) of Theorem 5.3 (page 112), for $x \to \infty$.

   d. Prove part (d) of Theorem 5.3 (page 112), for $x \to -\infty$.

5. Prove part (f) of Theorem 5.4 (page 114), for $x \to -\infty$.

6. Determine whether the conditionals listed below are true or false. If they are true prove it, if they are false give a counterexample.

   a. If $\lim_{x \to \infty} g(x) = -\infty$ and $\lim_{y \to -\infty} f(y) = \infty$, then

   $$\lim_{x \to \infty} f(g(x)) = \infty.$$

   b. If $\lim_{x \to -\infty} g(x) = \infty$ and $\lim_{y \to \infty} f(y) = -\infty$, then

   $$\lim_{x \to -\infty} f(g(x)) = -\infty.$$

7. Prove that

$$\lim_{x\to\infty} ax \pm b = \infty \quad \text{and} \quad \lim_{x\to-\infty} ax \pm b = -\infty$$

for any linear function $g(x) = ax \pm b$ where $a > 0$.

8. Prove that $\displaystyle\lim_{x\to-\infty} \frac{2}{(4x-1)\pi} = 0.$

9. Prove that $\displaystyle\lim_{x\to-\infty} \frac{6}{(12x-1)\pi} = 0.$

10. Evaluate the limit

$$\lim_{x\to\frac{1}{\pi}^-} \frac{\csc\left(\frac{1}{x}\right)}{x-\frac{1}{\pi}}.$$

11. Explain why L'Hôpital's rule cannot be applied to evaluate the limit

$$\lim_{x\to\frac{1}{\pi}^-} \frac{\csc\left(\frac{1}{x}\right)}{x-\frac{1}{\pi}} \quad \text{of Example 5.11 (page 114).}$$

Complete solutions are provided on page 314.

Chapter 6
Continuity in the Evaluation of Limits

It is said that a function is continuous if its graph does not "break."

Although this "visual" interpretation is considered to be "useful," it is incorrect and leads to misunderstandings.

The definition of a continuous function at *a number* is as follows.

Definition 6.1. A function f is *continuous at a number* a if

$$\lim_{x \to a} f(x) = f(a).$$

It is implicit in this definition that

- the number a is in the domain of f. That is, $f(a)$ is defined.
- $f(x)$ is defined for $x \to a$.
- the limit $\lim_{x \to a} f(x)$ exist and it is equal to $f(a)$.

Therefore, a function may only be continuous at a number a if and only if the number a is in its domain.

Negation of Definition 6.1.

A function f is discontinuous at a number a if either

- the number a is not in the domain of f, or
- the limit $\lim_{x \to a} f(x)$ does not exist, or
- the limit $\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$.

Remark 6.2. *By Definition 1.9 (page 15), a function f is continuous at a if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that*

$$|f(x) - f(a)| < \varepsilon \quad \textit{for all} \quad 0 < |x - a| < \delta. \tag{6.1}$$

Since a continuous function f is defined at this number a, the expression (6.1) is valid for $x = a$. Hence, a function f is continuous at a if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \textit{for every} \quad |x - a| < \delta. \tag{6.2}$$

Definition 6.3. A function f is *continuous everywhere* if

$$\lim_{x \to a} f(x) = f(a) \quad \text{for every} \quad x \in \mathbb{R}.$$

The functions given in the next three examples are not continuous everywhere.

Example 6.1. The domain of the function

$$f(x) = \frac{1}{x} \quad \text{is all nonzero numbers.}$$

Hence, it cannot be continuous at zero. The function is continuous on its domain since

$$\lim_{x \to a} \frac{1}{x} = \frac{1}{a} \quad \text{for any nonzero } a.$$

See Exercise 5 (page 71). □

Example 6.2. The domain of the function

$$g(x) = \begin{cases} x & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

is $\mathbb{R}$. It is not continuous at zero because

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x = 0 \neq 1 = g(0).$$

Hence, it is not continuous everywhere. □

Example 6.3. The domain of the function

$$h(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is $\mathbb{R}$. Since the limit

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist}$$

the function is not continuous at zero, and therefore it is not continuous everywhere. □

In the next example, we prove that the exponential function e^x is continuous everywhere.

Example 6.4. We apply (6.2) to prove that e^x is continuous for every number a.

For any positive number $\varepsilon > 0$ there is the positive number

$$\delta = \ln\left(\frac{\varepsilon}{e^a} + 1\right) > 0 \quad \text{such that if } |x - a| < \delta,$$

then

$$-\delta < x - a < \delta \Rightarrow -\delta < \ln(e^x) - \ln(e^a) < \delta.$$

By the properties of the natural logarithm

$$\ln(e^x) - \ln(e^a) = \ln\left(\frac{e^x}{e^a}\right).$$

Hence,

$$-\delta < \ln\left(\frac{e^x}{e^a}\right) < \delta.$$

Since the exponential function is increasing, we have

$$e^{-\delta} < \frac{e^x}{e^a} < e^{\delta} \quad \Rightarrow \quad e^{-\delta}e^a < e^x < e^{\delta}e^a$$

and

$$e^{-\delta}e^a - e^a < e^x - e^a < e^{\delta}e^a - e^a. \tag{6.3}$$

The right side of the inequality (6.3) is equal to

$$\begin{aligned} e^{\delta}e^a - e^a &= \left(\frac{\varepsilon}{e^a} + 1\right)e^a - e^a \\ &= \left(\frac{\varepsilon e^a}{e^a} + e^a\right) - e^a = \varepsilon + e^a - e^a = \varepsilon. \end{aligned}$$

The left side of this same inequality (6.3) is equal to

$$\begin{aligned} e^{-\delta}e^a - e^a &= \frac{e^a}{e^{\delta}} - e^a = \frac{e^a}{\dfrac{\varepsilon}{e^a} + 1} - e^a \\ &= \frac{e^a}{\dfrac{\varepsilon + e^a}{e^a}} - e^a = \frac{e^{2a}}{\varepsilon + e^a} - e^a \\ &= \frac{e^{2a} - e^a(\varepsilon + e^a)}{\varepsilon + e^a} = \frac{e^{2a} - \varepsilon e^a - e^{2a}}{\varepsilon + e^a} \\ &= \frac{-\varepsilon e^a}{\varepsilon + e^a} \end{aligned}$$

Since $e^a < \varepsilon + e^a$ we have that $\dfrac{e^a}{\varepsilon + e^a} < 1$. Hence,

$$\frac{-\varepsilon e^a}{\varepsilon + e^a} > -\varepsilon.$$

Therefore, from the inequality (6.3)

$$-\varepsilon < e^x - e^a < \varepsilon \quad \Rightarrow \quad |e^x - e^a| < \varepsilon.$$

$\square$

We must always specify where a function is continuous.

Examples of everywhere continuous functions are:

i. polynomials

ii. sine and cosine

iii. exponentials

iv. inverse trigonometric tangent and cotangent

v. hyperbolic sine, cosine, tangent, secant.

It is incorrect to say that a function is discontinuous because its graph "breaks."

If the graph of a function is "broken," then the function is not continuous everywhere, but it may well be contiguous on *its domain* as we see next.

The continuity of a function at a number from the right or left is as follows.

Definition 6.4. a. A function f is *continuous at the right of a number* a if

$$\lim_{x \to a^+} f(x) = f(a).$$

b. A function f is *continuous at the left of a number* a if

$$\lim_{x \to a^-} f(x) = f(a).$$

Again, a function may only be continuous from the right or left of a number if and only if this number is in its domain.

It is implicit in part (a) of Definition 6.4 that

- The function f is defined for $x \to a^+$, and
- the limit $\lim\limits_{x \to a^+} f(x)$ exist and it is equal to $f(a)$.

in part (b)

- The function f is defined for $x \to a^-$, and
- the limit $\lim\limits_{x \to a^-} f(x)$ exist and it is equal to $f(a)$.

Remark 6.5. *Similarly to Remark 6.2 (page 121), we have that a function f is continuous at the right of a if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that*

$$|f(x) - f(a)| < \varepsilon \quad \textit{for every} \quad 0 \leq a - x < \delta. \tag{6.4}$$

And, a function f is continuous at the left of a if for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - f(a)| < \varepsilon \quad \textit{for every} \quad 0 \leq x - a < \delta. \tag{6.5}$$

Definition 6.4 allows us to define continuous functions on their domains.

Definition 6.6. If the domain D_f of a function f is

a. an open interval, then f is continuous on its domain if

$$\lim_{x \to a} f(x) = f(a) \quad \text{for every } a \in D_f.$$

b. a semi-closed interval such as $[u, \infty)$ or $[u, v)$, then f is continuous on its domain if

$$\lim_{x \to u^+} f(x) = f(u)$$

and

$$\lim_{x \to a} f(x) = f(a) \quad \text{for every } u < a \text{ or } u < a < v \text{ respectively.}$$

c. a semi-closed interval such as $(-\infty, v]$ or $(u, v]$, then f is continuous on its domain if

$$\lim_{x \to v^-} f(x) = f(v)$$

and

$$\lim_{x \to a} f(x) = f(a) \quad \text{for every } a < v \text{ or } u < a < v \text{ respectively.}$$

d. a closed interval such as $[u, v]$, then f is continuous on its domain if

$$\lim_{x \to a} f(x) = f(a) \quad \text{for every } u < a < v,$$

$$\lim_{x \to u^+} f(x) = f(u) \quad \text{and} \quad \lim_{x \to v^-} f(x) = f(v).$$

If the domain D_f is the union of any type of intervals listed above, then the function f is continuous on D_f if it is continuous in each one of the intervals of its union.

Example 6.5. The domain of the function

$$\sin^{-1} x \quad \text{is the close interval } [-1, 1].$$

Hence, $\sin^{-1} x$ is continuous on its domain because

$$\lim_{x \to a} \sin^{-1} x = \sin^{-1} a \quad \text{for every } -1 < a < 1$$

and also

$$\lim_{x \to -1^+} \sin^{-1} x = \sin^{-1}(-1) = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x \to 1^-} \sin^{-1} x = \sin^{-1}(1) = \frac{\pi}{2}.$$

□

Example 6.6. The domain of the function

$$\tanh^{-1} x \quad \text{is the open interval } (-1, 1).$$

Hence, $\tanh^{-1} x$ is continuous on its domain because

$$\lim_{x \to a} \tanh^{-1} x = \tanh^{-1} a \quad \text{for} \quad -1 < a < 1.$$

□

Example 6.7. The domain of the function

$$h(x) = \sqrt{x} \quad \text{is the semi-open interval } [0, \infty).$$

We apply the statements (6.2) on page 121 and (6.4) on page 124 to prove that h is continuous in its domain.

Let $a > 0$ be in the domain of $f(x) = \sqrt{x}$. For any positive number $\varepsilon > 0$ there is $\delta = \varepsilon\sqrt{a} > 0$ such that if $|x - a| < \delta$, then

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{\sqrt{x} + \sqrt{a}} < \frac{\delta}{\sqrt{a}} = \varepsilon.$$

Hence,

$$\lim_{x \to a} \sqrt{x} = \sqrt{a} \quad \text{for every } a > 0.$$

On the other hand, for any positive number $\varepsilon > 0$ there is $\delta = \varepsilon^2 > 0$, such that if $0 < x < \delta$, then $\sqrt{x} < \sqrt{\varepsilon^2} = \varepsilon$. Hence,

$$\lim_{x \to 0^+} \sqrt{x} = \sqrt{0} = 0.$$

□

Examples of continuous functions on their domains are:

i. rational

ii. trigonometric tangent, cotangent, secant, and cosecant

iii. logarithmic

iv. inverse trigonometric sine, cosine, cotangent, secant, cosecant

v. hyperbolic cotangent and cosecant

vi. inverse hyperbolic.

Negation of Definition 6.4 (page 123)

a. A function f is discontinuous at a number a from the right if either

- the number a is not in the domain of f, or
- the limit $\lim_{x \to a^+} f(x)$ does not exist, or
- the limit $\lim_{x \to a^+} f(x)$ exists and it is not equal to $f(a)$.

b. A function f is discontinuous at a number a from the left if either

- the number a is not in the domain of f, or
- the limit $\lim_{x \to a^-} f(x)$ does not exist, or
- the limit $\lim_{x \to a^-} f(x)$ exists and it is not equal to $f(a)$.

Example 6.8. The domain of the function

$$h(x) = \begin{cases} \sqrt{x} & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \qquad \text{is the interval } [0, \infty).$$

This function is not continuous at zero from the right because

$$\lim_{x \to 0^+} h(x) = \lim_{x \to 0^+} \sqrt{x} = 0 \neq 1 = h(0).$$

Hence, it is not continuous on its domain. □

Example 6.9. The domain of the function

$$g(x) = \begin{cases} \frac{1}{x} & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x = 0 \end{cases} \qquad \text{is the interval } (-\infty, 0].$$

It is not continuous at zero from the left because the limit

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

is infinite, and therefore it does not exist. Hence, this function is not continuous on its domain. □

By the Laws of Limits (Theorem 3.2 on page 47) and Theorem 3.7 on page 63, we have the following theorem.

Theorem 6.7. *If the functions f and g are continuous at a number a, then the following functions are also continuous at the number a.*

a. $f(x) \pm g(x)$*,*

b. $f(x)g(x)$*,*

c. $Cf(x)$ *for any constant* C*,*

d. $\dfrac{f(x)}{g(x)}$ *if* $g(a) \neq 0$*, and*

This theorem also holds for continuous functions at the right and left of a number.

Proof. The proof is a direct consequence of Theorem 3.2 (page 47) because we have that

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a).$$

Thus,

a. $\lim_{x \to a}[f(x) \pm g(x)] = f(a) \pm g(a)$.

b. $\lim_{x \to a}[f(x)g(x)] = f(a)g(a)$.

c. $\lim_{x \to a}[cf(x)] = cf(a)$ for any constant c.

d. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f(a)}{g(a)}$ only if $g(a) \neq 0$.

Q.E.D.

Remark 6.8. *From Theorem 6.7 we conclude that if f and g are two functions continuous on their domains, then all the functions listed in this theorem are also continuous on their domains.* □

Example 6.10. By Exercise 4 of Chapter 3

$$\lim_{x \to a} x = a \quad \text{for any number } a.$$

Hence, by part (b) of Theorem 6.7

$$\lim_{x \to a} x^n = a^n \quad \text{for any positive integer } n.$$

By parts (a) and (c) of Theorem 6.7 we conclude that all polynomials functions are continuous everywhere. □

Example 6.11. The functions $f(x) = \sqrt{x}$ and $g(x) = \sin^{-1} x$ are continuous on their domains: $[0, \infty)$ and $[-1, 1]$ respectively. Thus, the functions

1. $f(x) + g(x) = \sqrt{x} + \sin^{-1} x$,
2. $f(x)g(x) = \sqrt{x} \sin^{-1} x$, and
3. $af(x) + bg(x) = a\sqrt{x} + b \sin^{-1} x$ where a, b are any numbers.

are continuous on $[0, \infty) \cap [-1, 1] = [0, 1]$.
Since $\sin^{-1}(0) = 0$, the function

$$\frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sin^{-1} x} \qquad \text{is continous on (0,1].}$$

$\square$

We stated in (3.24) on page 64 that if

$$\lim g(x) = b$$

and the function f is continuous at b, then

$$\lim f(g(x)) = f\left(\lim g(x)\right) = f(b) = c.$$

We prove this claim in the next Proposition.

Proposition 6.9. *If* $\lim g(x) = b$ *and* f *is continuous at* b*, then*

$$\lim f(g(x)) = f(b).$$

This proposition also holds for b^+ and b^-.

Proof. We give the proof for $x \to a$. The proofs for $x \to a^\pm$ and $x \to \pm\infty$ are similar. As in the proof of Theorem 3.7 (page 63). For any positive number $\varepsilon > 0$, there is a positive number $\delta_1 > 0$ such that

$$|f(y) - f(b)| < \varepsilon \quad \text{for every} \quad |y - a| < \delta_1.$$

For this positive number $\delta_1 > 0$, there is a positive number $\delta_2 > 0$ such that

$$|g(x) - b| < \delta_1 \quad \text{for every} \quad 0 < |x - a| < \delta_2.$$

Thus, for $y = g(x)$

$$|f(g(x)) - f(b)| < \varepsilon \quad \text{for every} \quad 0 < |x - a| < \delta_1.$$

Q.E.D.

Corollary 6.10. *If the function g is continuous at b and f is continuous at $g(b)$, then the composition $f \circ g(x)$ is continuous at b.*

This corollary holds for b^+ and b^-.

Proof. It follows directly from Proposition 6.9 since $\lim_{x \to b} g(x) = g(b)$. Q.E.D.

Remark 6.11. *By Corollary 6.10 we conclude that if g and f are two continuous functions on their domains, then its composition $f(g(x))$ is continuous on its domain.*

The function $f(x) = x^n$ for some positive integer n is continuous everywhere, then if g is continuous on its domain, then the composition $f(g(x)) = (g(x))^n$ is continuous on the domain of g.

Example 6.12. The functions $f(x) = x^n$ for some positive integer n and $g(x) = \ln x$, are continuous on their domains. Thus, the composition

$$f(g(x)) = (\ln x)^n \quad \text{is continuous on its domain which is } (0, \infty).$$

□

Example 6.13. The exponential function e^x and the linear function $f(x) = cx$ for any number c, are continuous everywhere Thus, its composition

$$e^{f(x)} = e^{cx} \quad \text{is also continuous everywhere.}$$

□

Continuity is also preserved for the inverse functions.

Theorem 6.12. *If f is a continuous and one-to-one function on an interval I, then its inverse f^{-1} is also continuous on the interval $f(I) = \{f(x) | \, x \in I\}$; the image of I under f.*

The proof of this theorem is beyond the scope of this manuscript. However, we encourage you to read its proof in [S] [1]

In Example 6.4 (page 122), we proved that the exponential function e^x is continuous everywhere. Its inverse function is the natural logarithm function $\ln x$. By Theorem 6.12, the function $\ln x$ is continuous on $(0, \infty)$ which is the range of e^x.

[1] Spivak, M. *Calculus*, (1994) Ed. Publish or Perish.

Any exponential function a^x of base a is continuous everywhere because for any $a > 0$

$$a^x = e^{cx} \quad \text{where} \quad c = \ln a,$$

and the exponential function e^{cx} is continuous everywhere (Example 6.13).

Similarly, any logarithmic function $\log_a x$ is continuous on its domain $(0, \infty)$ because for any $a > 0$

$$\log_a x = \frac{\ln x}{\ln a}.$$

See Theorem 6.7 (page 128).

Transformations

Basically, we study three types of transformations: shifts (S), compressions/stretches (T) and reflections (R). These everywhere continuous transformations are of fundamental importance in the evaluation of limits.

The effects that these transformations have on the graph of a function f are described below.

1. When a function f is composed with the *shift* transformation

   $$S_c(x) = x + c \quad (c > 0),$$

   its graph is shifted as follows.

   a. $S_c(f(x)) = f(x) + c$ c units **up**.
   b. $S_{-c}(f(x)) = f(x) - c$ c units **down**.
   c. $f(S_c(x)) = f(x + c)$ c units to the **left**.
   d. $f(S_{-c}(x)) = f(x - c)$ c units to the **right**.

2. When a function f is composed with the *stretch/compress* transformation

   $$T_c(x) = cx \quad (c > 1),$$

   its graph is stretched or compressed as follows.

   a. $T_c(f(x)) = cf(x)$ **stretch it vertically** by a factor of c units.

b. $T_{1/c}(f(x)) = \dfrac{f(x)}{c}$ **compress it vertically** by a factor of c units.

c. $f(T_c(x)) = f(cx)$ **compress it horizontally** by a factor of c units.

d. $f(T_{1/c}(x)) = f\left(\dfrac{x}{c}\right)$ **stretch it horizontally** by a factor of c units.

3. When a function f is composed with the *reflection* transformation

$$R(x) = -x,$$

its graph is reflected with respect to the axis as follows.

a. $R(f(x)) = -f(x)$ the x**-axis**.

b. $f(R(x)) = f(-x)$ the y**-axis**.

Since all these transformations are everywhere continuous, by Corollary 6.10 (page 130), their compositions with a continuous function f are also continuous on the domain D_f of f. That is, the graph of f does not "break" when f is composed with these transformations.

Students must be clear that shifts, compressions, stretches, and reflections are the results of compositions with these transformations.

When several transformations are composed with a function f, the effect on the graph of the function f is described from *right to left.*

Example 6.14. Let $f(x) = |x|$ and consider the composition

$$T_2 \circ R \circ f \circ S_{-2}(x) = T_2 \circ R(f(x-2)) = T_2(-f(x-2)) = -2f(x-2).$$

The effect on the graph of f is as in the following order.

1. For the composition

$$f \circ S_{-2}(x) = f(x-2) = |x-2|$$

the graph of f is shifted 2 units to the right.

2. For the composition

$$R \circ f \circ S_{-2}(x) = -f(x-2) = -|x-2|$$

the graph of $f(x-2)$ is reflected with respect to the x axis.

3. For the composition

$$T_2 \circ R \circ f \circ S_{-2}(x) = -2f(x-2) = -2|x-2|$$

the graph of $-f(x-2)$ is stretched vertically by 2 units. In summary, the graph of f is shifted 2 units to the right, followed by a reflection with respect to the x axis, followed by a vertical stretch by two units.

See Exercise 3 (page 154). □

Limits of Discontinuous Functions

We now turn out attention to the evaluation of limits of discontinuous functions.

We may apply Theorem 6.13 below to evaluate limits of a function f which is discontinuous at a, a^+ or a^-, and none of the results in Chapters 3, 4, and 5 apply.

Theorem 6.13. *If $f(x) = g(x)$ for $x \to a$, and*

$$\lim_{x\to a} g(x) \quad \textit{exist or is equal to } \infty \textit{ or } -\infty,$$

then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x).$$

This theorem also holds for $x \to a^\pm$ and $x \to \pm\infty$.

Proof. We give the proof for $x \to a$ and $\lim_{x\to a} g(a) = L$.

There is a positive number $\delta_1 > 0$, such that $f(x) = g(x)$ for all $0 < |x-a| < \delta_1$.

For any positive number $\varepsilon > 0$, there is a positive number $\delta_2 > 0$, such that

$$|g(x) - L| < \varepsilon \quad \text{for every} \quad 0 < |x-a| < \delta_2.$$

Hence, for $\delta = \min(\delta_1, \delta_1) > 0$

$$|f(x) - L| < \varepsilon \quad \text{for every} \quad 0 < |x-a| < \delta.$$

The proofs of the other cases are similar. Q.E.D.

Take note that Theorem 6.13 applies whenever it is possible to modify the function f

- algebraically,
- applying identities, or
- applying properties of the function f itself,

in such a way that the hypothesis of the theorem are met.

We illustrate the application of Theorem 6.13 (page 133) in the next examples.

Example 6.15. The function

$$f(x) = \ln(x-1) - \ln(x^2-1) \quad \text{is not continuous at 1 from the right.}$$

However, by the properties of the natural logarithm

$$\begin{aligned} f(x) &= \ln\left(\frac{x-1}{x^2-1}\right) = \ln\left(\frac{x-1}{(x-1)(x+1)}\right) \\ &= \ln\left(\frac{1}{x+1}\right) \\ &= -\ln(x+1) \quad \text{for every } x > 1. \end{aligned}$$

Thus,

$$f(x) = -\ln(x+1) \quad \text{for } x \to 1^+$$

and the function

$$g(x) = -\ln(x+1) \quad \text{is continuous at 1.}$$

By Theorem 6.13 (page 133),

$$\lim_{x\to 1^+} \ln(x-1) - \ln(x^2-1) = \lim_{x\to 1^+} -\ln(x+1) = -\ln(2).$$

□

Example 6.16. The function

$$f(x) = \sin\left(\frac{x+2}{2-3x-2x^2}\right) \quad \text{is not continuous at } -2.$$

However,

$$\frac{x+2}{2-3x-2x^2} = \frac{x+2}{(x+2)(x-1/2)} = \frac{1}{x-1/2} \quad \text{for all } x \neq -2, 1/2.$$

Thus,

$$f(x) = \frac{1}{x-1/2} \quad \text{for } x \to -2,$$

the function

$$g(x) = \frac{1}{x-1/2} \quad \text{is continuous at } -2$$

and the sine function is continuous everywhere.

By Theorem 6.13 (page 133), and Proposition 6.9 (page 129),

$$\begin{aligned}
\lim_{x\to -2} \sin\left(\frac{x+2}{2-3x-2x^2}\right) &= \lim_{x\to -2} \sin\left(\frac{1}{x-1/2}\right) \\
&= \sin\left(\lim_{x\to -2} \frac{1}{x-1/2}\right) \\
&= \sin\left(\frac{1}{-2-1/2}\right) = \sin\left(-\frac{2}{5}\right).
\end{aligned}$$

□

Limits at Infinity of Polynomial and Rational Functions

Let $P(x)$ and $Q(x)$ be the polynomial functions

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + ax + a_0 \quad Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + bx + b_0.$$

To evaluate the limits of polynomials and rational functions at infinity we apply Example 5.9. In this example we show that

$$\lim_{x\to\infty} P(x) = \begin{cases} \infty & \text{if} \quad a_n > 0 \\ -\infty & \text{if} \quad a_n < 0 \end{cases} \tag{6.6}$$

and

$$\lim_{x\to-\infty} P(x) = \begin{cases} -\infty & \text{if} \quad n \text{ is an odd positive integer and } a_n > 0 \\ \infty & \text{if} \quad n \text{ is an odd positive integer and } a_n < 0 \\ \infty & \text{if} \quad n \text{ is an even positive integer and } a_n > 0 \\ -\infty & \text{if} \quad n \text{ is an even positive integer and } a_n < 0 \end{cases} \tag{6.7}$$

In the same example we show that

$$\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m}.$$

If $m = n$, then

$$\lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x\to\pm\infty} \frac{a_n}{b_m} = \frac{a_n}{b_m}.$$

If $n < m$, then $m - n > 0$ and by Theorem 4.13 (page 94)

$$\lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x\to\pm\infty} \frac{a_n}{b_m x^{m-n}} = 0.$$

If $n > m$, then $n - m > 0$ and

$$\lim_{x\to\pm\infty} \frac{a_n x^n}{b_m x^m} = \lim_{x\to\pm\infty} \frac{a_n x^{n-m}}{b_m}.$$

We have several cases to consider.

By Example 5.1 (page 107) and Corollary 5.2 (page 106)

$$\lim_{x\to\infty} \frac{a_n x^{n-m}}{b_m} = \begin{cases} \infty & \text{if} \quad \frac{a_n}{b_m} > 0 \\ -\infty & \text{if} \quad \frac{a_n}{b_m} < 0 \end{cases}$$

$$\lim_{x\to-\infty} \frac{a_n x^{n-m}}{b_m} = \begin{cases} -\infty & \text{if} \quad n-m \text{ is an odd positive integer and } \frac{a_n}{b_m} > 0 \\ \infty & \text{if} \quad n-m \text{ is an odd positive integer and } \frac{a_n}{b_m} < 0 \\ \infty & \text{if} \quad n-m \text{ is an even positive integer and } \frac{a_n}{b_m} > 0 \\ -\infty & \text{if} \quad n-m \text{ is an even positive integer and } \frac{a_n}{b_m} < 0 \end{cases}$$

In summary,

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\begin{cases}\frac{a_n}{b_m} & \text{if} \quad m=n\\ 0 & \text{if} \quad n<m\\ \pm\infty & \text{if} \quad n>m\end{cases} \tag{6.8}$$

Example 6.17. In the limit

$$\lim_{x\to\infty}\cos\left(\frac{x+2}{2-3x-2x^2}\right)$$

the function

$$h(x)=\cos\left(\frac{x+2}{2-3x-2x^2}\right)$$

is the composition of the cosine and the rational function

$$\frac{x+2}{2-3x-2x^2}.$$

Since,

$$\lim_{x\to\infty}\frac{x+2}{2-3x-2x^2}=0,$$

by Theorem 5.4 (page 114)

$$\lim_{x\to\infty}\cos\left(\frac{x+2}{2-3x-2x^2}\right)=\cos(0)=1.$$

□

Trigonometric Limits

We apply Theorem 6.13 (page 133) to evaluate several trigonometric limits.

The evaluation of the limit

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{6.9}$$

can be found in most calculus textbooks, see for example any edition of [JS].[2]

From the limit (6.9) we deduce the *trigonometric limits* listed in the proposition below.

[2] Stewart, J. *Single Variable Calculus*, Brooks/Cole.

Proposition 6.14. *Let*

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Hence, for any nonzero numbers u, v

a. $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x} = 0.$

b. $\displaystyle\lim_{x\to 0} \frac{\sin(ux)}{x} = u.$

c. $\displaystyle\lim_{x\to 0} \frac{x}{\sin(ux)} = \frac{1}{u}.$

d. $\displaystyle\lim_{x\to 0} \frac{\sin(ux)}{\sin(vx)} = \frac{u}{v}.$

e. $\displaystyle\lim_{x\to 0} \frac{\tan(ux)}{\tan(vx)} = \frac{u}{v}.$

Proof. We apply Theorem 6.13 (page 133).

a. By the trigonometric identity $\sin^2 x = 1 - \cos^2 x$, we have

$$\begin{aligned} \frac{1-\cos x}{x} &= \left[\frac{1-\cos x}{x}\right]\left[\frac{1+\cos x}{1+\cos x}\right] \\ &= \frac{1-\cos^2 x}{(1+\cos x)x} = \frac{\sin^2 x}{x(1+\cos x)} \\ &= \left[\frac{\sin x}{x}\right]\left[\frac{\sin x}{1+\cos x}\right] \quad \text{for } x \to 0. \end{aligned}$$

By the Laws of Limits (Theorem 3.2 on page 47)

$$\lim_{x\to 0} \frac{\sin x}{1+\cos x} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0}(1+\cos x)} = \frac{0}{2} = 0.$$

and by Theorem 6.13 (page 133),

$$\begin{aligned} \lim_{x\to 0} \frac{1-\cos x}{x} &= \lim_{x\to 0} \left[\frac{\sin x}{x}\right]\left[\frac{\sin x}{1+\cos x}\right] \\ &\left[\lim_{x\to 0} \frac{\sin x}{x}\right]\left[\lim_{x\to 0} \frac{\sin x}{1+\cos x}\right] \\ &= 1(0) = 0. \end{aligned}$$

b. It is true that

$$\frac{\sin(ux)}{x} = \frac{u(\sin(ux))}{ux} \quad \text{for } x \to 0.$$

By Corollary 3.3 (page 53)

$$\lim_{x\to 0} ux = u(0) = 0.$$

If $t = ux$, then $t \to 0$ as $x \to 0$.
By part (c) of Theorem 3.7 (page 63) and the limit (6.9)

$$\lim_{x\to 0} \frac{u\sin(ux)}{ux} = \lim_{t\to 0} u\left(\frac{\sin t}{t}\right) = u(1) = u. \tag{6.10}$$

c. This limit follows from part (b) of this proposition and part (d) of Theorem 3.2 (page 47).

d. We have

$$\frac{\sin(ux)}{\sin(vx)} = \left[\frac{\sin(ux)}{x}\right]\left[\frac{x}{\sin(vx)}\right] \quad \text{for } x \to 0.$$

By the Laws of Limits (Theorem 3.2 on page 47) and part (c) of this proposition

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(ux)}{\sin(vx)} &= \lim_{x\to 0} \left[\frac{\sin(ux)}{x}\right]\left[\frac{x}{\sin(vx)}\right] \\ &\left[\lim_{x\to 0} \frac{\sin(ux)}{x}\right]\left[\lim_{x\to 0} \frac{x}{\sin(vx)}\right] \\ &= \frac{u}{v}.\end{aligned}$$

e. We have

$$\frac{\tan(ux)}{\tan(vx)} = \left[\frac{\sin(ux)}{\sin(vx)}\right]\left[\frac{\cos(vx)}{\cos(ux)}\right]$$

By the continuity of the cosine function at zero

$$\lim_{x\to 0} \frac{\cos(vx)}{\cos(ux)} = \frac{\cos(0)}{\cos(0)} = 1.$$

By Theorem 3.2 (page 47) and part (c) of this proposition

$$\begin{aligned}\lim_{x\to 0} \frac{\tan(ux)}{\tan(vx)} &= \left[\lim_{x\to 0} \frac{\sin(ux)}{\sin(vx)}\right]\left[\lim_{x\to 0} \frac{\cos(vx)}{\cos(ux)}\right] \\ &= \frac{u}{v}\end{aligned}$$

Q.E.D.

Example 6.18. To evaluate the limit

$$\lim_{x\to 3}\frac{\sin(x-3)}{3-x},$$

we apply Theorem 3.7 (page 63). By Corollary 3.3 (page 53), if $t = x - 3$, then

$$t \to 0 \quad \text{as } x \to 3.$$

Hence,

$$\lim_{x\to 3}\frac{\sin(x-3)}{3-x} = \lim_{t\to 0} -\frac{\sin t}{t} = -1.$$

$\square$

Example 6.19. To evaluate the limit

$$\lim_{x\to 0}\frac{x}{1-\cos x}$$

we apply Theorem 4.13 (page 94) to the limits

$$\lim_{x\to 0^+}\frac{x}{1-\cos x} \quad \text{and} \quad \lim_{x\to 0^-}\frac{x}{1-\cos x}.$$

We know that $\cos x < 1$ for $x \to 0$ (see Figure 1.4 on page 6).
Thus, $1 - \cos x > 0$ for $x \to 0$ and

$$\frac{1-\cos x}{x} > 0 \quad \text{for } 0 < x < \frac{\pi}{2}.$$

By 3.6 (page 61),

$$\lim_{x\to 0^+}\frac{1-\cos x}{x} = 0^+.$$

Hence, by part (a) of Theorem 6.13 (page 133) and Theorem 4.13 (page 94),

$$\lim_{x\to 0^+}\frac{x}{1-\cos x} = \infty.$$

Similarly,

$$\frac{1-\cos x}{x} < 0 \quad \text{for } -\frac{\pi}{2} < x < 0$$

and by Proposition 3.6 (page 61),

$$\lim_{x\to 0^-} \frac{1-\cos x}{x} = 0^-.$$

Hence, by part (a) of Theorem 6.13 (page 133) and Theorem 4.13 (page 94),

$$\lim_{x\to 0^-} \frac{x}{1-\cos x} = -\infty.$$

Therefore, by the negation of Proposition 4.8 (86), the limit

$$\lim_{x\to 0} \frac{x}{1-\cos x} \quad \text{does not exist.}$$

$\square$

L'Hospital's Rule

The application of L'Hospital's Rule [3] must be carefully understood. It only applies to the type of limits listed in Definition 6.15.

Definition 6.15. Let f and g be two functions

a. If $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$, then the limit

$$\lim \frac{f(x)}{g(x)} \quad \text{is of type } \infty/\infty.$$

b. If $\lim f(x) = 0$ and $\lim g(x) = 0$, then the limit

$$\lim \frac{f(x)}{g(x)} \quad \text{is of type } 0/0.$$

c. If $\lim f(x) = 0$ and $\lim g(x) = \pm\infty$, then the limit

$$\lim f(x)g(x) \quad \text{is of type } 0\infty.$$

d. If $\lim f(x) = 0$ and $\lim g(x) = \pm\infty$, then the limit

$$\lim g(x)^{f(x)} \quad \text{is of type } \infty^0.$$

[3] This rule is also spelled as l'Hôpital

e. If $\lim f(x) = 1$ and $\lim g(x) = \pm\infty$, then the limit

$$\lim f(x)^{g(x)} \quad \text{is of type } 1^{\infty}.$$

f. If $\lim f(x) = 0$ and $\lim g(x) = 0$, then the limit

$$\lim f(x)^{g(x)} \quad \text{is of type } 0^{0}.$$

Students must know how to evaluate limits in order to determine whether a limit is of any of the types listed in Definition 6.15.

Theorem 6.16. ***L'Hospital's Rule.*** *If the limit*

$$\lim \frac{f(x)}{g(x)} \quad \textit{is of type} \;\; 0/0 \;\textit{or}\;\; \infty/\infty,$$

then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

if the limit on the right exist or is equal to ∞ *or* $-\infty$*.*

The proof of L'Hospital's Rule can be found in [S] [4].

It is implicit in Theorem 6.16 that

- both functions f and g are differentiable on their domains.
- $g'(x) \neq 0$ wherever the limit $\lim \dfrac{f'(x)}{g'(x)}$ is defined.

Limits of type 0∞.

If either

$$\lim f(x) = 0^{+} \quad \text{or} \quad \lim f(x) = 0^{-}$$

and

$$\lim g(x) = \pm\infty,$$

[4] Spivak, M. *Calculus*, (1994) by Publish or Perish.

then by Theorem 4.13 and Remark 4.14 (pages 94 and 96, respectively)

$$\lim \frac{1}{g(x)} = 0^{\pm} \tag{6.11}$$

and

$$\lim \frac{1}{f(x)} = \pm\infty \tag{6.12}$$

If (6.11) and (6.12) hold, then we may be able to apply l'Hospital's rule to the limits

$$\lim \frac{f(x)}{1/g(x)} \quad \text{and} \quad \lim \frac{g(x)}{1/f(x)} \tag{6.13}$$

because the former is of type $0/0$ and the latter is of type ∞/∞.

Example 6.20. To evaluate the limit

$$\lim_{x\to\infty} x \sin\left(\frac{1}{x}\right) \tag{6.14}$$

we consider the limits

$$\lim_{x\to\infty} x = \infty, \quad \text{[by (6.6) on page 135]}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0^+, \quad \text{[by Theorem 4.13 on page 94]}$$

and

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = \lim_{t\to 0^+} \sin t = 0^+ \quad \text{[by Proposition 6.9 (page 129)]}$$

Hence, the limit (6.14) is of type 0∞ and we can apply l'Hospital's rule. Since the limit

$$\lim_{x\to\infty} \frac{1}{\sin\left(\frac{1}{x}\right)} \qquad \text{is undefined (see Figure 1.11 (page 17)),}$$

we must consider the limit

$$\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{1/x} \quad \text{of type } 0/0. \tag{6.15}$$

The derivatives of the numerator and denominator functions are:

$$\frac{d}{dx} \sin\left(\frac{1}{x}\right) = -\cos\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right)$$

and

$$\frac{d}{dx}\,\frac{1}{x} = -\frac{1}{x^2}.$$

Applying l'Hospital's rule we have by Proposition 6.9 (page 129),

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\,\sin\left(\frac{1}{x}\right)}{\frac{d}{dx}\,\frac{1}{x}} = \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1.$$

Therefore the limit (6.14) is equal to 1. □

Study the next example carefully. Pay attention to the importance that the choice of the limits (6.13) have in order to apply l'Hospital's rule properly, and the strategy used to evaluate the given limit.

Example 6.21. To evaluate the limit

$$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right)\ln x \tag{6.16}$$

We recall that

$$\lim_{x\to 0^+} x\sin\left(\frac{1}{x}\right) = 0^+ \quad \text{(see Example 3.4 (page 60))}.$$

Since $\lim\limits_{x\to 0^+} \ln x = -\infty$, the limit (6.16) is of type 0∞.

To apply l'Hospital's rule we must consider the quotient

$$\frac{x\sin\left(\frac{1}{x}\right)}{1/\ln x} \tag{6.17}$$

because the reciprocal of the numerator function

$$\frac{1}{x\sin\left(\frac{1}{x}\right)} \quad \text{is undefined for } x\to 0^+ \text{ as shown in Example 4.9 (page93).}$$

If we take the derivatives of the numerator and denominator of the quotient (6.17), we have

$$\frac{d}{dx}\left[x\sin\left(\frac{1}{x}\right)\right] = \sin\left(\frac{1}{x}\right) + x\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

and

$$\frac{d}{dx}\left[\frac{1}{\ln x}\right] = -\frac{\frac{1}{x}}{(\ln x)^2} = -\frac{1}{x(\ln x)^2}.$$

Thus, we end up with the limit

$$\lim_{x\to 0^+} -\frac{(\ln x)^2}{x}\left[x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)\right].$$

We cannot apply l'Hospital's rule because this limit is none of the types listed in Definition 6.15.

We change our approach and consider the product of $x \ln x$ and $\sin\left(\frac{1}{x}\right)$ instead.

The function $x \ln x < 0$ for $x \to 0^+$; thus,

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1 \Rightarrow -x\ln x \ge (x\ln x)\sin\left(\frac{1}{x}\right) \ge x \ln x$$

We must evaluate the limits on the right and left of this inequality to apply the Squeeze Theorem 3.13 on page 70. The limit

$$\lim_{x\to 0^+} x\ln x \quad \text{is of type } 0\infty \text{ [why?].}$$

We apply l'Hospital's rule to the quotient of $\ln x$ and $1/x$

$$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} -\frac{1/x}{1/x^2} = \lim_{x\to 0^+} -x = 0.$$

By the Squeeze Theorem

$$\lim_{x\to 0^+} (x\ln x)\sin\left(\frac{1}{x}\right) = 0.$$

What happens if we apply l'Hospital's rule to the limit

$$\lim_{x\to 0^+} \frac{x}{1/\ln x}?$$

□

Remark 6.17. *If*

$$\lim f(x) = 0 \quad \textit{neither from the right nor left}$$

and

$$\lim g(x) = \pm\infty,$$

we must apply l'Hospital's rule to the limit of type $0/0$

$$\lim \frac{f(x)}{1/g(x)} \quad \textit{if it is defined.}$$

Limits of type ∞^0, 1^∞ and 0^0.

If we apply the natural logarithmic function to the function $h(x) = f(x)^{g(x)}$, we get

$$\ln(h(x)) = \ln\left(f(x)^{g(x)}\right) = g(x)\ln(f(x)).$$

Thus,

$$\lim \ln(h(x)) = \lim g(x)\ln(f(x)). \tag{6.18}$$

The limit (6.18) is of type 0∞, if

- $\lim f(x) = \infty$ and $\lim g(x) = 0$,
- $\lim f(x) = 1$ and $\lim g(x) = \pm\infty$, or
- $\lim f(x) = 0^+$ and $\lim g(x) = 0$.

In other words, the limit (6.18) is of type 0∞ if and only if the limit

$$\lim f(x)^{g(x)} \quad \text{is of type } \infty^0,\ 1^\infty \text{ or } 0^0.$$

The exponential function e^x is continuous everywhere and

$$\lim_{x\to\infty} e^x = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} e^x = 0.$$

Thus, if the limit

- $\lim g(x)\ln(f(x)) = K$ exists, then by Proposition 6.9 (page 129),

$$\lim f(x)^{g(x)} = \lim \exp[g(x)\ln(f(x))] = \exp[\lim g(x)\ln(f(x))] = e^K. \tag{6.19}$$

- $\lim g(x)\ln(f(x)) = \infty$, then by part (f) of Theorem 5.4 (114),

$$\lim f(x)^{g(x)} = \lim \exp[g(x)\ln(f(x))] = \infty. \tag{6.20}$$

- $\lim g(x)\ln(f(x)) = -\infty$, then by part (e) of Theorem 5.4 (page 114),

$$\lim f(x)^{g(x)} = \lim \exp[g(x)\ln(f(x))] = 0. \tag{6.21}$$

Remark 6.18. *For these types of limits, we must verify that the limit (6.18) is defined. Observe that the function $\ln(x^2)$ is defined for all non-zero x, and the function $2\ln x$ is defined for all $x > 0$. Hence,*

$$\ln(x^2) = \ln(|x|^2) = 2\ln|x| \qquad \textit{for } x \neq 0.$$

and

$$\ln(x^2) = 2\ln x \qquad \textit{for } x > 0.$$

Example 6.22. The limit

$$\lim_{x\to 0^-} (\sin^2 x)^x \quad \text{is defined since } \sin^2 x \geq 0 \text{ for every number } x.$$

If we apply the natural logarithm to the function $(\sin^2 x)^x$ we may consider

$$\ln(\sin^2 x)^x = x\ln(\sin x)^2 = 2x\ln(\sin x).$$

However, $\sin x < 0$ for $x \to 0^-$ and therefore $\ln(\sin x)$ is undefined for $x \to 0^-$.
So we must consider $2x\ln|\sin x|$ instead of $2x\ln(\sin x)$.
Thus,

$$\ln|\sin x| = \ln(-\sin x) \quad \text{for } x \to 0^-.$$

We apply l'Hospital's rule to the limit

$$\lim_{x\to 0^-} \frac{2\ln(-\sin x)}{1/x} \quad \text{of type } \infty/\infty.$$

We have,

$$\lim_{x\to 0^-} \frac{\frac{-2\cos x}{\sin x}}{-1/x^2} = \lim_{x\to 0^-} \frac{2x^2\cos x}{\sin x}.$$

This limit of type $0/0$ and we apply l'Hospital's rule once more together with the Laws of Limits.

$$\lim_{x\to 0^-} 2x\left[\frac{2\cos x - x\sin x}{\cos x}\right] = \left[\lim_{x\to 0^-} 2x\right]\left[\lim_{x\to 0^-} \frac{2\cos x - x\sin x}{\cos x}\right] = 0(2) = 0.$$

By (6.19)

$$\lim_{x\to 0^-} (\sin^2 x)^x = e^0 = 1.$$

□

Example 6.23. We consider the well-defined limit

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x.$$

We apply the natural logarithm function to the function $\left(\frac{1}{x}\right)^x$.

$$\ln\left(\frac{1}{x}\right)^x = x\ln\left(\frac{1}{x}\right) = -x\ln x \quad \text{for } x\to 0^+.$$

Thus, the limit

$$\lim_{x\to 0^+} -x\ln x \quad \text{is of type } 0\infty.$$

We apply l'Hospital's rule to the limit

$$\lim_{x\to 0^+} -\frac{\ln x}{1/x} = \lim_{x\to 0^+} -\frac{1/x}{-1/x^2} = \lim_{x\to 0^+} x = 0.$$

Hence, by (6.19)

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^x = e^0 = 1.$$

□

Example 6.24. The limit

$$\lim_{x\to\infty}\left(\frac{1}{x}\right)^x \qquad \text{is of type } 0^\infty.$$

Thus, l'Hospital's rule does not apply. However, we can apply the same strategy and we apply the natural logarithm function to the function $\left(\frac{1}{x}\right)^x$.

$$\ln\left(\frac{1}{x}\right)^x = -x\ln x \quad \text{for } x\to\infty.$$

By Theorem 3.2 (page 47)

$$\lim_{x\to\infty} -x\ln x = -\infty.$$

By (6.21)

$$\lim_{x\to\infty}\left(\frac{1}{x}\right)^x = 0.$$

□

Limits of Type $\infty - \infty$

L'Hospital's rule may be used to evaluate limits of type $\infty - \infty$, only if the function can be modified in such a way that the resulting limit is of either type $0/0$ or ∞/∞. However, this is not always possible, as we show in the next example.

Example 6.25. The limit

$$\lim_{x\to 0^+} \frac{1}{x} + \ln x \quad \text{is of type } \infty - \infty.$$

We change it algebraically to get a quotient

$$\frac{1}{x} + \ln x = \frac{1 - x\ln x}{x}$$

But the limit

$$\lim_{x\to 0^+} \frac{1 - x\ln x}{x} \quad \text{is neither type } 0/0 \text{ nor } \infty/\infty.$$

We cannot apply l'Hospital's rule, but by Example 6.24

$$\lim_{x\to 0^+} -x\ln x = 0$$

and

$$\lim_{x\to 0^+} 1 - x\ln x = 1 \quad \text{by the Laws of Limits.}$$

Since,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

By part (e) of Theorem 5.1 (page 101)

$$\lim_{x\to 0^+} \frac{1 - x\ln x}{x} = \lim_{x\to 0^+} (1 - x\ln x)\left(\frac{1}{x}\right) = \infty.$$

□

Limits of Type $\infty/0$

To evaluate limits of type $\infty/0$ we must consider the quotient function $\dfrac{f(x)}{g(x)}$ as the product $f(x)\left(\dfrac{1}{g(x)}\right)$.

If $\lim g(x) = 0^+$, then by Theorem 4.13 (page 94),

$$\lim \frac{1}{g(x)} = \infty.$$

Therefore, if

$$\lim f(x) = \infty \quad \text{and} \quad \lim \frac{1}{g(x)} = \infty,$$

then by part (g) of Theorem 5.1 (page 101) we have that

$$\lim \frac{f(x)}{g(x)} = \lim f(x)\left(\frac{1}{g(x)}\right) = \infty.$$

Similar applications of Theorem 4.13 (page 94), and parts (g), (h) and (i) of Theorem 5.1 (page 101), give the following result.

Theorem 6.19. *If*

$$\begin{aligned} &\lim f(x) = \infty, \qquad && \lim F(x) = -\infty,\\ &\lim g(x) = 0^+, && \lim G(x) = 0^-, \end{aligned}$$

then

a. $\displaystyle \lim \frac{f(x)}{g(x)} = \infty.$

b. $\displaystyle \lim \frac{f(x)}{G(x)} = -\infty.$

c. $\displaystyle \lim \frac{F(x)}{g(x)} = -\infty.$

d. $\displaystyle \lim \frac{F(x)}{G(x)} = \infty.$

The limits listed in Theorem 6.19 must be defined in order to conclude. See that the limit

$$\lim_{x\to\infty} \sin\left(\frac{1}{x}\right) = 0^+$$

but the limit

$$\lim_{x\to\infty} \frac{x}{\sin(1/x)} \quad \text{is undefined, see Example 1.4 on page 9.}$$

Also, by Exercise 6 (page 99), the limit

$$\lim_{x\to 0^+} \frac{\ln x}{\omega(x)} \quad \text{is well-defined and of type } \infty/0$$

where $\omega(x)$ is the function defined in Example 4.8 (page 92). However, by part (b) of Exercise 12 (page 99)

$$\lim_{x\to 0^+} \frac{1}{\omega(x)} \quad \text{does not exist.}$$

Therefore, if

$$\lim_{x\to 0^+} \frac{\ln x}{\omega(x)} = K \quad \text{for some } K,$$

then by Theorem 3.2 (page 47)

$$0 = \left[\lim_{x\to 0^+} \frac{\ln x}{\omega(x)}\right]\left[\lim_{x\to 0^+} \omega(x)\right] = \lim_{x\to 0^+} \left(\frac{\ln x}{\omega(x)}\right)\omega(x) = \lim_{x\to 0^+} \ln x = -\infty.$$

This contradiction tells us that

$$\lim_{x\to 0^+} \frac{\ln x}{\omega(x)} \quad \text{does not exist.}$$

Thus, it cannot be infinite. See Exercise 8 of this chapter.

Summary

In Figure 6.1, we present a Venn diagram indicating the relations of finite limits, infinite limits and non-existing limits.

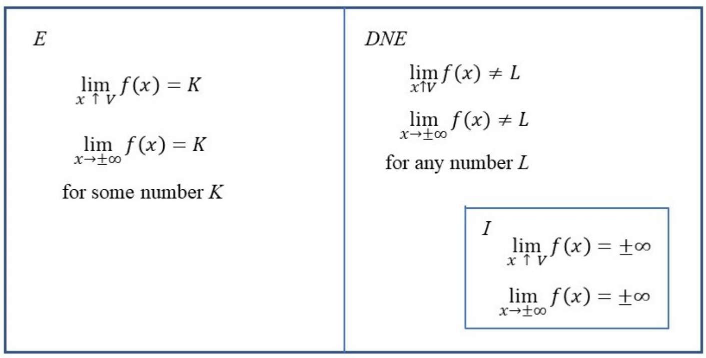


Figure 6.1: Venn diagram of finite, non-existing and infinite limits

The set E consists of finite limits, the set DNE consists of limits which do not exist, and the set I consists of infinite limits.

The sets E and DNE are disjoint. That is, a limit either exists or does not exist but not both.

The set I is a proper subset of the set DNE. That is, all infinite limits do not exist, and there are limits which do not exist and are not infinite.

Looking at the complements of these sets.

- If a limit is not finite, then either it does not exist or it is infinite, or both.
- If a limit is not infinite, then either it is finite or it does not exist but not both.

We presented the sequence of learning of limits from their definitions and their negations.

1. **Finite Limits.** Chapters 1 - 3.

   (a) **Finite limits at a number.** Definition 1.9 (page 15), Definition 1.10 (page 20), Definition 1.11 (page 22).

(b) **Finite limits at infinity.** Definition 2.5 (page 35), Definition 2.6 (page 40). Horizontal asymptotes. Definition 2.7 (page 43).

(c) **Properties of finite limits.** Laws of Limits (Theorem 3.2 on page 47).

i. Finite Limits from the right and left. Definition 3.4 (page 56).

ii. Limits of polynomial and rational functions. Corollary 3.3 (page 53).

iii. Finite Limits of composition of functions. Theorem 3.7 (page 63).

iv. Finite limits of inequalities. The Squeeze Theorem (Theorem 3.13 on page 70).

2. **Infinite Limits** Chapters 4 - 5.

(a) **Infinite limits at a number.** Definition 4.4 (page 81), Definition 4.5 (page 82). Vertical asymptotes. Definition 4.6 (page 83).

(b) **Infinite limits at infinity.** Definition 4.9 (page 89).
Reciprocal functions and finite side limits. Theorem 4.13 (page 94)

(c) **Properties of infinite limits.** Theorem 5.1 (page 101), Corollary 5.2 (page 106)

i. Limits of polynomial functions at infinity. Example 5.9 (page 110).

ii. Inequalities. Theorem 5.3 (page 112).

iii. Composition of functions. Theorem 5.4 (page 114).

3. **Evaluation of Limits** Chapter 6.

(a) Continuity. Definition 6.1 (page 120).

(b) Continuous functions on their domains. Definition 6.6 (page 124).

(c) Transformations. Chapter 6 (page 131).

(d) Limits of discontinuous functions. Theorem 6.13 (page 133).

(e) Limits of polynomial functions at infinity. Equation (6.7) (page 136).

(f) Limits of rational functions at infinity. Equation (6.8) (page 137).

(g) Squeeze Theorem. Theorem 3.13 (page 70).

(h) Trigonometric limits. Proposition 6.14 (page 138).

(i) L'Hospital's rule. Definition 6.15 (page 141).

(j) Limits of type $\infty - \infty$. Theorem 6.19 (page 150).

(k) Limits of Type $\infty/0$. Theorem 6.19 (page 150).

(l) Limits of Type $0/\infty$. Exercise 9.

Exercises

1. a. By Exercise 9 of Chapter 1 the sine function is continuous on the close interval $\left[0, \frac{\pi}{2}\right]$. Apply transformations to prove that the sine function is everywhere continues.
   b. Use part (a) to prove that the cosine function is everywhere continuous.

2. Sketch the graph of the composition

   $$T_2 \circ R \circ f \circ S_{-2}(x)$$

   of Example 6.14 (page 132).

3. Describe the transformations and the order in which they modify the graph of the function

   $$f(x) = \begin{cases} 1 & if \quad x < 0 \\ -1 & if \quad 0 < x \end{cases}$$

   in the composition

   $$S_2 \circ T_{1/3} \circ R \circ f(x).$$

   Give the graph of the function $S_2 \circ T_{1/3} \circ R \circ f(x)$.

4. Give an example of one and only one function f which satisfies all the conditions listed below.

   a. The domain of f is $\mathbb{R}$.
   b. It is discontinuous at 0.
   c. It is continuous from the left at 0.
   d. It is continuous on the interval $(0, \infty)$.

5. Give an example of a rational function $R(x)$ so that

   a. $\lim_{x\to\infty} R(x) = \frac{2}{3}$,
   b. $R(1) = 0 = R(3)$, and
   c. $R(-1)$ is undefined.

6. Determine the type of the limits listed below.

   a. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x}$

   b. $\lim\limits_{x\to \infty} \dfrac{\sin x}{x}$

   c. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\sin x}$

7. Evaluate the limits listed below

   a. $\lim\limits_{x\to 0} \dfrac{1-\sec x}{x}$

   b. $\lim\limits_{x\to 0} \dfrac{\tan(ux)}{x}$ for any nonzero u.

   c. $\lim\limits_{x\to 0} \dfrac{x}{\tan(ux)}$ for any nonzero u.

8. Let $\omega(x)$ be the function defined in **??** (page **??**). In Exercise 6 of Chapter 4 it was proved that

   $$\lim_{x\to 0} \omega(x) = 0.$$

   Prove that

   $$\lim_{x\to 0} \omega(x) = 0 \quad \text{from neither the left nor right of zero.}$$

9. If

   $$\lim f(x) = 0 \quad \text{and} \quad \lim g(x) = \pm\infty,$$

   then

   $$\lim \frac{f(x)}{g(x)} \quad \text{is of type } 0/\infty.$$

   Apply Remark 4.14 (page 96) to explain why in this case

   $$\lim \frac{f(x)}{g(x)} = 0.$$

Complete solutions are provided on page 323.

Chapter 7
Teaching Limits

In Chapters 1-6, we cover all necessary definitions and results about limits and their evaluation. Calculus instructors must master the content of these chapters.

In the next three chapters, we provide our recommendations on how to teach limits when definitions of limits are left out, left as optional, or learned after the material of limits have been covered.

At the end of each section, we state the learning objectives followed by suggested exercises. Always look for or make up your own exercises, aiming, at a minimum, to test each objective.

Basic Functions

If the definition of a limit is not our starting point, we must provide a solid base to develop our understanding of what a limit is.

The best approach is to introduce a minimum number of "basic" everywhere continuous functions, such as

i. Constant function $C(x) = c$.

ii. Identity function $I(x) = x$.

iii. Simplest square function $S(x) = x^2$.

iv. Simplest cubic function $C(x) = x^3$.

v. Absolute value function $V(x) = |x|$.

vi. Since and cosine functions.

These functions are taken as "axioms" from which we can generate more continuous functions by composing them with the transformations described in Chapter 6 (page 131). Moreover, students should be able to sketch the graphs of the compositions of basic functions and transformations, by applying the effect that the transformations have on the graphs of these basic functions.

Since continuity is preserved under these transformations, all the functions listed below, where b and c are any positive real numbers, are examples of continuous everywhere functions.

a. $S_b(T_c(x)) = cx + b$ and $S_{-b}(T_c(x)) = cx - b$ linear functions,

b. $S_b(T_c(S(x))) = cx^2 + b$ and $S_{-b}(T_c(S(x))) = cx^2 - b$ quadratic functions,

c. $C_c(S(S_b(x))) = c(x + b)^2$ and $C_c(S(S_{-b}(x))) = c(x - b)^2$ quadratic functions,

d. $S_b(T_c(C(x))) = cx^3 + b$ and $S_{-b}(T_c(C(x))) = cx^3 - b$ cubic functions,

e. $C_c(C(S_b(x))) = c(x + b)^3$ and $C_c(C(S_{-b}(x))) = c(x - b)^3$ cubic functions,

f. $S_b(T_c(\sin x)) = c \sin x + b$ and $S_{-b}(T_c(\sin x)) = c \sin x - b$,

g. $S_b(T_c(\cos x)) = c \cos x + b$ and $S_{-b}(T_c(\cos x)) = c \cos x - b$, and

h. $\sin(S_b(x)) = \sin(x + b)$ and $\sin(C_c(x)) = \sin(cx)$.

At this point, students do not know what a continuous function is, but they should be able to sketch the graphs of these basic functions and their composition with the transformations described in Chapter 6 (page 131).

Piecewise functions are very useful as examples and counterexamples. Hence, students must learn

- to identify their domains,
- how to evaluate them, and
- sketch their graphs.

Objectives

Students should be able to

1. Sketch the graphs of all the basic functions.
2. compose functions.

3. know the effect that transformations have on the graphs of basic functions.
4. sketch the graphs of composition of functions with transformations.
5. find the domains of piecewise functions.
6. evaluate piecewise functions.
7. sketch the graphs of piecewise functions.

Each of the questions below tests the objectives of this section in sequential order. Review them and make up your own questions.

Exercises

1. Give the composition functions indicated below.
   a. $F_1(F_4(x))$ where $F_1(x) = x + 4$ and $F_4(x) = \dfrac{x}{4}$.
   b. $\sin(C(x))$ where $C(x) = x^3$.
   c. $F_3(S(F_2(x)))$ where $F_3(x) = 2x$, $F_2(x) = x - \dfrac{2}{3}$, and $S(x) = x^2$.
2. Describe the effect that the compositions listed below have on the graph of a function f.
   a. $T_{1/2}(S_3(f(x)))$ where $T_{1/2}(x) = \dfrac{x}{2}$ and $S_3(x) = x + 3$.
   b. $f(T_2(x))$ where $T_2 = 2x$.
   c. $S_{-4}((f(R(x)))$ where $S_{-4}(x) = x - 4$, $R(x) = -x$.
3. Sketch the graphs of the functions listed in the previous question where $f(x) = \cos x$.

4. Give a piecewise function with domain all real numbers except the numbers 0, 2, and $\dfrac{5}{2}$.
5. Evaluate the piecewise function

$$f(x) = \begin{cases} \sin x & \text{if} \quad x < -\frac{\pi}{2} \\ x^2 & \text{if} \quad -\frac{\pi}{2} < x \le 3 \\ x^3 - 1 & \text{if} \quad \pi \le x < 2\pi \end{cases}$$

at the numbers $-\dfrac{\pi}{4}, 3, 7$.

Explain why $f\left(\dfrac{9\pi}{2}\right)$ is undefined.

6. Sketch the graph of the function in the previous question.

Complete solutions are provided on page 329.

Visual Understanding of Values of a Function

We teach limits at

- a number from the right ($x \to a^+$),
- a number from the left ($x \to a^-$),
- a number ($x \to a$), and
- infinity ($x \to \pm\infty$).

We strongly recommend to teach them in this order.

As it was explained in Chapter 1, the understanding of "closeness" of two numbers is fundamental in the learning of limits.

To arrive at the *intuitive ideas* of limits at a number a from the right or left, we must know

a. which numbers are at the left or right of a number a (Definition 1.1 on page 1),

b. that there are numbers sufficiently close to a number a from the right or left (Remark 1.7 on page 11),

c. when a number is very close to a number a from the left or right (Definition 1.2 on page 3), and

d. how to locate the values $f(x)$ of a function f for numbers x sufficiently close to a number a from the right or left.

To consider numbers from the right, left and around a number a, students should get used to working with open intervals of the forms $(a, a+\delta)$, $(a-\delta, a)$ and $(a-\delta, a+\delta)$ for some positive number $\delta > 0$. The use of these intervals makes easier for the students to understand that the smaller δ the closer a number is to the number a.

We know that

> there are numbers sufficiently close to a number a from the right or left

because *any* open interval is nonempty.
Hence, for any number a and any positive number $\delta > 0$, there is a number between a and $a+\delta$ and a number between $a-\delta$ and a.

Visually, the interval $(a, a+\delta)$ is the set of all numbers between a and $a+\delta$.

Figure 7.1: There is a number x in the open interval $(a, a+\delta)$

Since the number $\delta > 0$ is arbitrary, it can be taken arbitrary *small*, making the distance $x - a > 0$ between the numbers a and x very small. Thus, there are numbers very close to a from the right.

The interval $(a - \delta, a)$ is the set of all numbers between $a - \delta$ and a.

Figure 7.2: There is a number x in the open interval $(a - \delta, a)$

Again, since the number $\delta > 0$ is arbitrary small, there are numbers very close to a from the left.

We conclude that there are no "holes" on the real line. Therefore, we know that there are numbers sufficiently close to a number a from the right or left.

Based on this fact we say that

- $x \to a^+$ represents the set of those numbers very close to the number a from the right. That is,

$$x \to a^+ = \{x \in \mathbb{R} \mid a < x < a + \delta \quad \text{for a very small } \delta > 0.\}$$

- $x \to a^-$ represents the set of those numbers very close to the number a from the left. That is,

$$x \to a^- = \{x \in \mathbb{R} |\ a - \delta < x < a \quad \text{for a very small } \delta > 0.\}$$

- $x \to a$ represents the set of those numbers close to the number a from the right and left. That is,

$$x \to a = \{x \in \mathbb{R} |\ a - \delta < x < a + \delta \quad \text{for a very small } \delta > 0.\}$$

See Definition 1.2 (page 3).

The next step is to visually locate the values of $f(x)$ for numbers x sufficiently close to the number a from the right and left.

In the next three examples, we identify the domains and values of the piecewise functions close to a number.

Example 7.1. The domain of the piecewise function

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ x & \text{if} \quad x \geq 2 \end{cases} \qquad \text{is } \mathbb{R}.$$

Its graph is shown in Figure 7.3.

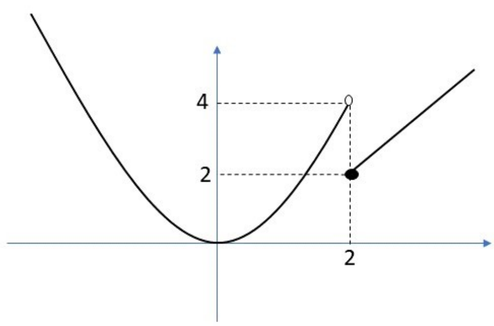


Figure 7.3: The graph of the piecewise $f(x)$

The values of this function around 2 are

- $f(2) = 2$,
- $f(x) = x^2 \quad$ for $x \to 2^-$,
- $f(x) = x \quad$ for $x \to 2^+$.

$\square$

Example 7.2. The domain of the piecewise function

$$g(x) = \begin{cases} x^2 & \text{if} \quad x < 2 \\ x & \text{if} \quad x \geq 3 \end{cases} \qquad \text{is } \mathbb{R} - [2,3) = (-\infty, 2) \cup [3, \infty).$$

Its graph is shown in Figure 7.4.

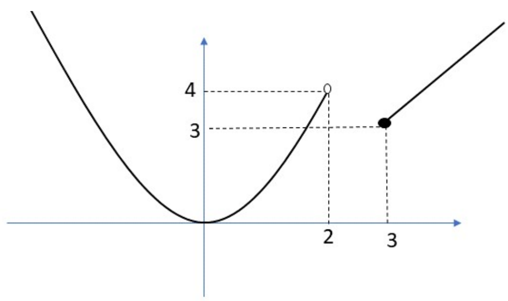


Figure 7.4: The graph of the piecewise $g(x)$

This function is undefined for numbers sufficiently close to 2 form the right. That is, the function is undefined for $x \to 2^+$. It is also undefined for numbers sufficiently close to 3 from the left. That is, the function is undefined for $x \to 3^-$.

The values of this function around the number 2 are

$$g(x) = x^2 \quad \text{for } x \to 2^-.$$

The values of this function around the number 3 are

$$g(3) = 3 \quad \text{and} \quad g(x) = x \quad \text{for } x \to 3^+.$$

□

Example 7.3. The graph of a piecewise function $h(x)$ is shown in Figure 7.5.

Its domain is the union of three intervals: $(-\infty, a) \cup [b, 0) \cup (0, \infty)$.

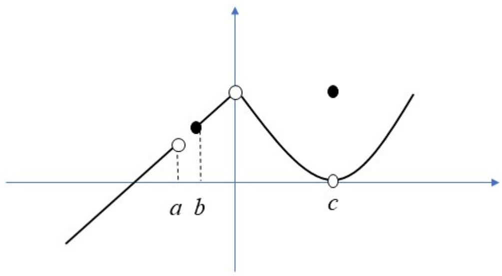


Figure 7.5: The graph of the piecewise $h(x)$

It is undefined at the numbers a and 0, and for $x \to a^+$ and $x \to b^-$.

It is defined for $x \to a^-$, $x \to b^+$, $x \to 0$ and $x \to c$. $\square$

In Examples 7.1 to 7.3, we pay attention to the values of the functions on the x-axis .

In Figure 7.3, we pay attention to the numbers, on the x-axis, close to the number 2 from the right. The corresponding values $f(x)$ of f on the y-axis are close to the number 2 from the right. That is,

$$f(x) - 2 > 0 \quad \text{for } x \to 2^+.$$

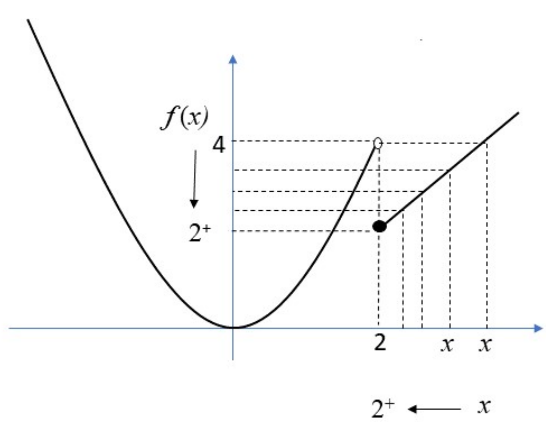


Figure 7.6: $f(x)$ is close to 2 as $x \to 2^+$

In Figure 7.4, we consider the numbers, on the x-axis, close to the number 2 from the left. The corresponding values $g(x)$ of g on the

y-axis are close to the number 4 from the left. That is,

$$4 - g(x) > 0 \quad \text{for } x \to 2^-.$$

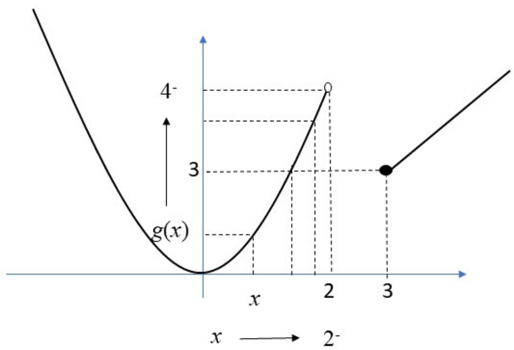


Figure 7.7: $g(x)$ is close to 4 as $x \to 2^-$

In Figure 7.5, we consider the numbers, on the x-axis, close to the number c from the left and right. The corresponding values $h(x)$ of h on the y-axis are close to the number 0 from the right. That is,

$$h(x) > 0 \quad \text{for } x \to c.$$

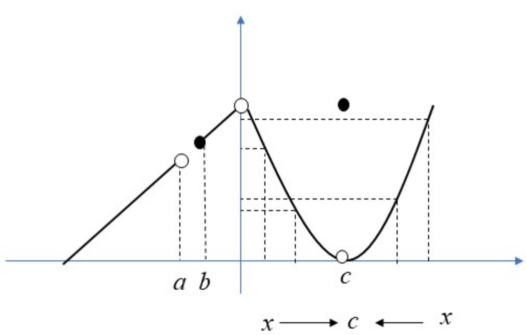


Figure 7.8: $h(x)$ is close to 0 as $x \to c$

Using particular cases, we explain what we mean by the values of a function to be close to a number and to be close from the right and left.

The values $f(x)$ of a function f are close to a number L, if their distance is small (see the distance between two numbers on page 3).

A. The values of $f(x)$ are very close to a number L from the right if

$$0 < f(x) - L < \varepsilon \quad \text{and } \varepsilon > 0 \text{ is a very small number.}$$

B. The values of $f(x)$ are very close to a number L from the left if

$$0 < L - f(x) < \varepsilon \quad \text{and } \varepsilon > 0 \text{ is a very small number.}$$

C. The values of $f(x)$ are very close to a number L if

$$|f(x) - L| < \varepsilon \quad \text{and } \varepsilon > 0 \text{ is a very small number.}$$

We introduce next the notation of the situations described above.

- $f(x) \to R^{+}$
  represents those values $f(x)$ of the function f which are very close to the number R from the right,
- $f(x) \to L^{-}$
  represents those values $f(x)$ of the function f which are very close to the number L from the left, and
- $f(x) \to M$
  represents those values $f(x)$ of the function f which are very close to the number M from the right and left.

Students should do as many exercises as possible to visually locate the values of functions on the y-axis.

Objectives

Students should be able to

1. Determine how close two numbers are on the real line.

2. Locate the values of a function for numbers close to a number from the left and right.

3. Visually determine the values of a function at a number from the left and right.

Each of the questions below tests the objectives of this section. Review them and make up your own questions.

Exercises

7. If $\delta = 0.01$, give a number in the intervals listed below

   a. $(0, \delta)$

   b. $(2 - \delta, 2)$

   c. $(1.001, 1.001 + \delta)$

8. How close is π from 3.1416.

9. A. Sketch the graph of the function g

$$g(x) = \begin{cases} 3x + 2 & \text{if} \quad x < -2 \\ x & \text{if} \quad -2 < x < 2\pi \\ 2\sin x & \text{if} \quad x \geq 2\pi \end{cases}$$

   B. Give the values of the function at the numbers listed below.

   a. $g(-3)$

   b. $g(0)$

   c. $g(1)$

   d. $g(2\pi)$

   e. $g(3\pi)$

C. Give the number R so that

a. $g(x) \to R \quad$ for $x \to -2^-$
b. $g(x) \to R \quad$ for $x \to -2^+$
c. $g(x) \to R \quad$ for $x \to 2\pi^-$
d. $g(x) \to R \quad$ for $x \to 2\pi^+$

10. Give the graph of one and only one function f which satisfies all the conditions listed below.

    a. its domain is $\mathbb{R}$
    b. $f(x) \to -2^+$ as $x \to 2^-$
    c. $f(x) \to -2^+$ as $x \to 2^+$
    d. $f(x) \to 0^-$ as $x \to 3^+$
    e. $f(x) \to 1$ as $x \to 3^-$

Complete solutions are provided on page 332.

Intuitive Understanding of Limits

The intuitive ideas of side limits must be introduced only after the concept of "closeness" introduced on pages 161 and 165 is *well* understood.

We introduce the notation of limits at the *same* time we learn about numbers and values of a function being close.

We are ready to establish the *intuitive* meaning of side limits and limits. See statements (1.1) on page 15, (1.11) on page 20 and (1.14) on page 22.

Limit at the right of a number. If we can make the values of $f(x)$ to be very close to R (as close as we like) by taking x sufficiently close to a from the right, then

$$\lim_{x \to a^+} f(x) = R.$$

Limit at the left of a number. If we can make the values of $f(x)$ to be very close to L (as close as we like) by taking x sufficiently close to a from the left, then

$$\lim_{x \to a^-} f(x) = L.$$

Limit at a number. If we can make the values of $f(x)$ to be very close to M (as close as we like) by taking x sufficiently close to a, then

$$\lim_{x \to a} f(x) = M.$$

It is implicit in the limits stated above that

a. the number a may not be in the domain of f.

b. for $x \to a^+$, the domain of f must contain an open interval $(a, a + \delta)$ for some $\delta > 0$.

c. for $x \to a^-$, the domain of f must contain an open interval $(a - \delta, a)$ for some $\delta > 0$.

d. for $x \to a$, the domain of f must contain an open interval $(a - \delta, a) \cup (a, a + \delta)$ for some $\delta > 0$.

A limit is undefined if the conditions (a) - (c) are not met.

In Examples 7.1 to 7.3 we see that we can consider $f(x)$ for $x \to 2$ and $g(x)$ for $x \to 2^-$ and for $x \to 3^+$ but not for $x \to 2^+$ nor $x \to 3^-$. Thus,

- $\lim_{x \to 2} f(x)$ is well defined.
- $\lim_{x \to 2^-} g(x)$ is well defined.
- $\lim_{x \to 3^+} g(x)$ is well defined.
- $\lim_{x \to 2^+} g(x)$ is undefined.
- $\lim_{x \to 3^-} g(x)$ is undefined.

It must be made clear to the students that a limit

$$\lim_{x \to \uparrow a} f(x) \quad \text{does not exist}$$

if and only if

$$\lim_{x \to \uparrow a} f(x) \neq K \quad \text{for any number } K.$$

The relationship between side limits and limits at a number was established in Proposition 1.12 (page 24). We re-state it here for clarity.

Proposition 7.1. *The limit* $\lim_{x \to a} f(x) = M$ *exists* *if and only if*

a. $\lim_{x \to a^+} f(x) = R$ *exists,*

b. $\lim_{x \to a^-} f(x) = L$ *exists, and*

c. $R = L = M$.

This proportion says that

$$\lim_{x \to a} f(x) = M \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = M = \lim_{x \to a^-} f(x).$$

For example, in Figure 7.8 we see that

$$\lim_{x \to c^-} h(x) = 0 = \lim_{x \to c^+} h(x).$$

Thus, by Proposition 7.1

$$\lim_{x \to c} h(x) = 0.$$

In Figure 7.6 we see that

$$\lim_{x \to 2^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 2.$$

Thus, by Proposition 7.1

$$\lim_{x \to 2} f(x) \quad \text{does not exist.}$$

To understand non-existing limits, students must learn the negations of the statements above.

Negation of a limit at the right of a number. The limit

$$\lim_{x \to a^+} f(x) \qquad \text{does not exist if}$$

$$\lim_{x \to a^+} f(x) \neq R \qquad \text{for any number } R.$$

That is, for any number R, there is a number x close to the number a from the right, such that $f(x)$ is not close to the number R (i.e. the distance between $f(x)$ and R is not small).

Negation of a limit at the left of a number. The limit

$$\lim_{x \to a^+} f(x) \qquad \text{does not exist if}$$

$$\lim_{x\to a^-} f(x) \neq L \qquad \text{for any number } L.$$

That is, for any number L, there is a number x close to the number a from the left, such that $f(x)$ is not close to L (i.e. the distance between $f(x)$ and L is not small).

Negation of a limit at of a number. The limit

$$\lim_{x\to a^+} f(x) \qquad \text{does not exist if}$$

$$\lim_{x\to a} f(x) \neq M \qquad \text{for any number } M.$$

That is, for any number M, there is a number x close to the number a, such that $f(x)$ is not close to M (i.e. the distance between $f(x)$ and M is not small).

Students must also understand these negations visually.

For example, from the graph of the function

$$\sin\left(\frac{1}{x}\right) \qquad \text{Figure 1.11 on page 17,}$$

we see that for any number M, we can find a number x close to the number 0 (from the left or right) such that $f(x)$ is not close to M. That is, the distance between $f(x)$ and M is not small for some number x close to the number 0. Hence,

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist.}$$

Without the graph of a function, it is difficult or impossible for the students to explain why a limit does not exist. However, they must learn the negation of Proposition 7.1.

The limit

$$\lim_{x\to a} f(x) \quad \text{does not exist,}$$

if and only if either

a. $\lim\limits_{x\to a^-} f(x)$ does not exists,

b. $\lim\limits_{x\to a^+} f(x)$ does not exists, or

c. $\lim\limits_{x\to a^-} f(x) = L \neq R = \lim\limits_{x\to a^+} f(x)$.

Most students are capable of applying part (c) of this negation to explain why a limit does not exist.

Students should do as many exercises as possible to understand, visually, the existence and non-existence of limits.

Objectives

Students should be able to

1. Determine whether a limit is either well defined or undefined.
2. Locate the values of a function for numbers close to a number from the left and right.
3. Explain the value of a limit of a function by looking at its graph.
4. Explain the non-existence of a limit of a function by looking at its graph.

Each of the questions below tests the objectives of this section. Review them and make up your own questions.

Exercises

11. Sketch the graph of one and only one function f such that

    a. $\lim\limits_{x\to -1^+} f(x)$ is defined.

    b. $\lim\limits_{x\to -1^-} f(x)$ is undefined.

    c. $f(0)$ is defined and $\lim\limits_{x\to 0} f(x)$ is undefined.

12. Consider the graph of the function h shown below.

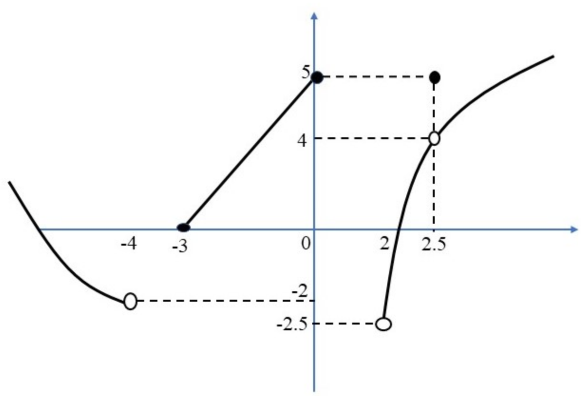


Figure 7.9: Graph of the function h

A. Give a number a such that $h(a)$ is undefined. Explain.

B. Give a number b such that $h(b)$ is defined and $\lim\limits_{x\to b} f(x) \neq h(b)$.

C. Explain why $\lim\limits_{x\to 0^+} h(x)$ is undefined.

D. Give a number $c \neq 0$ such that $\lim\limits_{x\to c^+} h(x)$ is undefined

E. Give the value of the limits listed below

   (a) $\lim\limits_{x\to -3^+} h(x)$,

(b) $\lim_{x \to 0^-} h(x)$,

(c) $\lim_{x \to 2^+} h(x)$,

Complete solutions are provided on page 335.

Chapter 8
Continuous Functions

Since students do not learn the definition of limits, they do not have to prove which functions are continuous. It is enough for them to see and accept from the basic functions' graphs that these functions are everywhere continuous; hence, their compositions with the transformations listed on page 131 are also everywhere continuous.

We start by defining continuity at a point to generalize it later to intervals. Again, appealing to our visual understanding of limits.

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a). \tag{8.1}$$

For (8.1) to hold we must have three things

1. The number a belongs to the domain of f; thus, $f(a)$ is defined,
2. the limit $\lim_{x \to a} f(x)$ exists, and
3. the limit $\lim_{x \to a} f(x) = f(a)$ exists and it is equal to $f(a)$.

Students must understand that if a function is continuous at a number a, then *visually*, its graph "does not break" around a. Hence, all basic functions are continuous at any number and therefore they are continuous *everywhere*.

A function f is not continuous at a number a if the negations of either of the three statements above do not hold. That is,

1. The number a does not belong to the domain of f; thus, $f(a)$ is not defined, or
2. the limit $\lim_{x\to a} f(x)$ does not exists, or
3. the limit $\lim_{x\to a} f(x)$ exists but it is not equal to $f(a)$.

In Figure 8.1, the number a is not in the domain of f, and the limit

$$\lim_{x\to a} f(x) = L \quad \text{exists.}$$

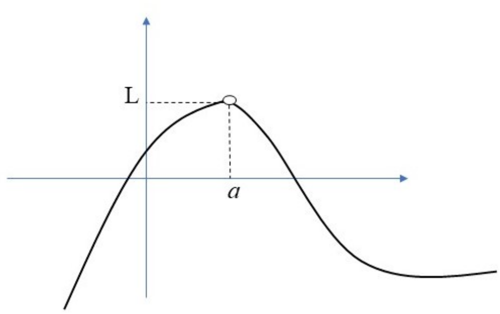


Figure 8.1: The function f is not defined at a number a

In Figure 8.2, the number a is in the domain of f, and the limit

$$\lim_{x\to a} f(x) \quad \text{does not exist.}$$

In Figure 8.3, the number a is in the domain of f, and the limit

$$\lim_{x\to a} f(x) = L \neq f(a) \quad \text{exists but it is not equal to } f(a).$$

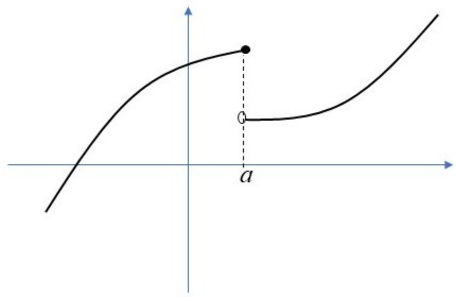


Figure 8.2: The limit $\lim_{x \to a} f(x)$ does not exists

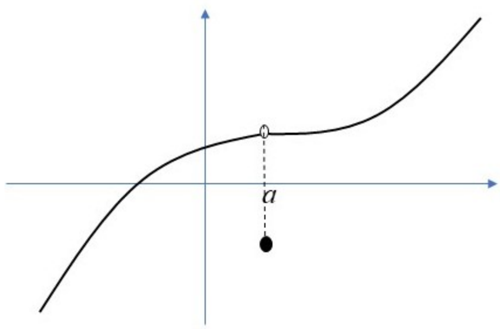


Figure 8.3: The limit $\lim_{x \to a} f(x)$ exists but it is not equal to $f(a)$

Next, we define the continuity at a number from the right and left.

A function f is continuous at a number a from the right if

$$\lim_{x \to a^+} f(x) = f(a). \tag{8.2}$$

For (8.2) to hold we must have three things

1. The number a belongs to the domain of f; thus, $f(a)$ is defined,
2. the limit $\lim_{x \to a^+} f(x)$ exists, and
3. the limit $\lim_{x \to a^+} f(x) = f(a)$ exists and it is equal to $f(a)$.

As before, a function f is not continuous at a number a from the right, if either

1. The number a does not belong to the domain of f; thus, $f(a)$ is not defined, or
2. the limit $\lim\limits_{x \to a^+} f(x)$ does not exists, or
3. the limit $\lim\limits_{x \to a^+} f(x)$ exists but it is not equal to $f(a)$.

A function f is continuous at a number a from the left if

$$\lim_{x \to a^-} f(x) = f(a). \tag{8.3}$$

For (8.3) to hold we must have three things

1. The number a belongs to the domain of f; thus, $f(a)$ is defined,
2. the limit $\lim\limits_{x \to a^-} f(x)$ exists, and
3. the limit $\lim\limits_{x \to a^-} f(x) = f(a)$ exists and it is equal to $f(a)$.

A function f is not continuous at a from the left, if either

1. The number a does not belong to the domain of f; thus, $f(a)$ is not defined, or
2. the limit $\lim\limits_{x \to a^-} f(x)$ does not exists, or
3. the limit $\lim\limits_{x \to a^-} f(x)$ exists but it is not equal to $f(a)$.

The function in Figure 8.2 (page 162), is continuous at a number a from the left, because $\lim_{x \to a^+} f(x) = f(a)$.

It is not continuous at a number a from the right, because $\lim_{x \to a^+} f(x) \neq f(a)$.

Below we present the generalization of continuity on intervals.

A function f is continuous on the interval

- $\mathbb{R} = (-\infty, \infty)$, if $f(a)$ is continuous for every number a. In this case, the function is continuous everywhere.
- $(-\infty, n)$, if $f(a)$ is continuous for every number $a < n$.
- (m, ∞), if $f(a)$ is continuous for every number $m < a$.
- $(-\infty, n]$, if $f(a)$ is continuous for every number $a < n$ and f is continuous at n from the left.
- $[m, \infty)$, if $f(a)$ is continuous for every number $m < a$ and f is continuous at m from the right.
- (m, n), if $f(a)$ is continuous for every number $m < a < n$.
- $[m, n)$, if $f(a)$ is continuous for every number $m < a < n$, and f is continuous at m from the right.
- $(m, n]$, if $f(a)$ is continuous for every number $m < a < n$, and f is continuous at n from the left.
- $[m, n]$, if $f(a)$ is continuous for every number $m < a < n$, and f is continuous at m from the right and at n from the left.

The function in Figure 8.1 (page 177), is continuous on the intervals $(-\infty, a)$ and (a, ∞).

The function in Figure 8.2 (page 178), is continuous on the intervals $(-\infty, a]$ and (a, ∞).

The function in Figure 8.3 (page 178), is continuous on the intervals $(-\infty, a)$ and (a, ∞).

In the next example, we explain where the given piecewise function is continuous.

Example 8.1. Let f be the function

$$f(x) = \begin{cases} (x+2)^2 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ -\frac{4x}{3} + 4 & \text{if} \quad 0 < x < 3 \\ 0 & \text{if} \quad 3 \leq x \leq 5 \\ -1 & \text{if} \quad x > 5 \end{cases}$$

The domain of f is $\mathbb{R}$ [Why?]. The function

$$f_1(x) = (x+2)^2 \quad \text{is the left shift of } x^2 \text{ by 2 units.}$$

The function

$$f_2(x) = -\frac{4x}{3} + 4 \quad \text{is a linear function.}$$

The functions

$$f_3(x) = 0, \;\; f_4(x) = -1 \quad \text{are constant functions.}$$

All these three functions are continuous everywhere. Hence the function f is continuous on the intervals $(-\infty, 0)$, $(0, 3)$, $(3, 5)$ and $(5, \infty)$ [Why?].

To determine whether the function f is continuous on $\mathbb{R}$, we need to consider the continuity of f at the numbers 0, 3, and 5.[Why?]

The side limits at zero are:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x+2)^2 = (0+2)^2 = 4$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} -\frac{4x}{3} + 4 = -\frac{0}{3} + 4 = 4.$$

Hence,

$$\lim_{x \to 0^+} f(x) = 4 \neq f(0) = 0 \quad \text{and the function } f \text{ is not continous at zero.}$$

The side limits at 3 are:

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} -\frac{4x}{3} + 4 = -\frac{4(3)}{3} + 4 = 0$$

and

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 0 = 0.$$

Hence,

$$\lim_{x \to 3} f(x) = 0 = f(3) \quad \text{and the function } f \text{ is continuous at 3.}$$

The side limits at 5 are:

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} 0 = 0 = f(5),$$

the function f is continuous at 5 from the left.

$$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} -1 = -1 \neq f(5).$$

Hence,

$$\lim_{x \to 5} f(x) \quad \text{does not exist and the function } f \text{ is not continous at 5.}$$

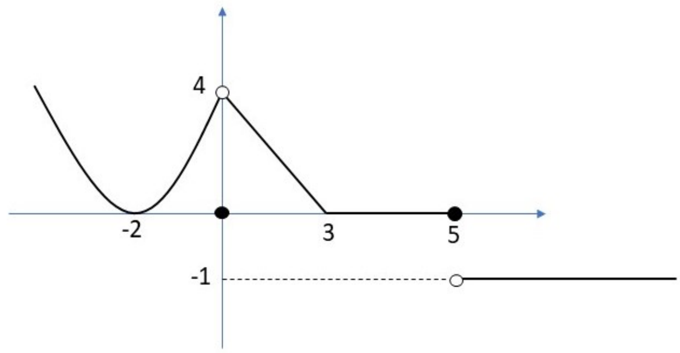


Figure 8.4: The graph of $f(x)$

We corroborate these statements, visually, by looking at the graph of f. We conclude that the function f is continuous on $(-\infty, 0) \cup (0, 5] \cup (5, \infty)$. $\square$

It is time to determine which functions are continuous on their domains.

We have established, visually, that the basic functions, listed on page 156, and their composition with the transformations listed on page 131 are continuous everywhere.

At this point, we do not have the Laws of Limits (Theorem 3.2 on page 47) yet, so we must recourse to our intuition and accept, for the moment, that the sum and product of continuous functions are continuous.

Theorem 8.1. *If*

$$\lim_{x\to a} f(x) = f(a) \quad \textit{and} \quad \lim_{x\to a} g(x) = g(a),$$

then

a. $\lim_{x\to a} f(x) + g(x) = f(a) + g(a)$

b. $\lim_{x\to a} f(x)g(x) = f(a)g(x)$

Theorem 8.1 says that if the functions f and g are continuous at a number a, then the number a is in their domains, and their sum $f+g$ and product fg are also continuous at the number a.

Therefore, if the function f and g are continuous on their domains D_f and D_g respectively, then their sum $f+g$ and product fg are continuous on the intersection $D_f \cap D_g$ of their domains.

The identity function $I(x) = x$ is continuous everywhere; hence, by part (b) of Theorem 8.1 (page 183), the polynomial function

$$P_n(x) = (I(x))^n = \underbrace{I(x)\ldots I(x)}_{n \text{ times}} = x^n$$

is continuous everywhere for any positive integer n.

Also, the composition

$$\mathcal{T}_c(P_n(x)) = cx^n$$

of the function $P_n(x)$ and the transformation $\mathcal{T}_c(x)$ is continuous everywhere, where $\mathcal{T}_c(x) = cx$ is either one of the transformations $C_c(x) = cx^n$ $(c < 1)$ or $T_c(x) = cx$ $(c > 1)$.

Note. For $n = 0$, we have $P_0(x) = x^0 = 1$; hence, the composition

$$T_c(P_0(x)) = c \quad \text{is the constant function } c.$$

Finally, by part (a) of Theorem 8.1 (page 183), any polynomial function

$$\begin{aligned} P(x) &= T_{a_n}(P_n(x)) + T_{a_{n-1}}(P_{n-1}(x)) + T_{a_{n-2}}(P_{n-2}(x)) \\ &\quad + \cdots + T_{a_1}(P_1(x)) + T_{a_0}(P_0(x)) \\ &= a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} \ldots a_1 x + a_0 \end{aligned}$$

is continuous everywhere.

Example 8.2. Explain why the functions listed below are everywhere continuous.

a. $f_1(x) = 3x + 6\cos x$

b. $f_2(x) = \sin x - x\cos^2 x + 7$

c. $f_3(x) = (5x^3 - 1)(3x + 6\cos x)$

d. $f_4(x) = |4x + 1|\sin^4 x$

They are everywhere continuous, because they are combinations of sums, products or compositions of basic functions and transformations.

a. $f_1(x) = 3x + 6\cos x = T_3(x) + T_6(\cos x)$

b. $f_2(x) = \sin x - x\cos^2 x + 7 = \sin x + I(x)P_2(\cos x) + \mathcal{T}_7(P_0(x))$

c. $f_3(x) = (5x^3 - 1)(3x + 6\cos x) =$
$[T_5(P_3(x)) - P_0(x)][T_3(x) + T_3(\cos x)]$

d. $f_4(x) = |4x + 1|\sin^4 x = V(T_4(x)) + P_0(x)$

$\square$

For the composition of continuous functions we have the theorem below, which is proved as Corollary 6.10 (page 130).

Theorem 8.2. *If the function g is continuous at b and f is continuous at $g(b)$, then the composition $f \circ g(x)$ is continuous at b.*

Theorem 8.2 says that if

$$\lim_{x \to b} g(x) = g(b) = a \quad \text{and} \quad \lim_{y \to a} f(y) = f(a),$$

then,

$$\lim_{x \to b} f(g(x)) = f\left[\lim_{x \to b} g(x)\right] = f(g(b)) = f(a).$$

Example 8.3. The function

$$f(x) = \sin\left(x - \frac{\pi}{3}\right) \quad \text{is everywhere continuous}$$

because it is the composition of the transformation $S_{\pi/3}(x)$ and the basic function $\sin x$. Hence,

$$\lim_{x\to\pi} \sin\left(x - \frac{\pi}{3}\right) = \sin\left(\lim_{x\to\pi} x - \frac{\pi}{3}\right) = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

$\square$

Students should always identify the set *where* a function is continuous.

The basic functions and their composition with transformations together with Theorem 8.1 and Theorem 8.2 provide us with infinitely many everywhere continuous functions.

Objectives

Students should be able to

1. Give examples of everywhere continuous functions.
2. Determine, using side limits, whether a function is continuous on its domain.
3. Determine, visually, whether a function is continuous on its domain.
4. Explain why a function is continuous on a given set.
5. Explain why a function is not continuous on a given set.
6. Sketch the graph of a continuous function on a given set.
7. Apply Theorem 8.2 (185), to determine whether the sum, product and/or composition of continuous functions is continuous on a given set.

Each of the questions below tests the objectives of this section. Review them and make up your own questions.

Exercises

1. If a function f is everywhere continuous and the function g is continuous on its domain D_g, where is their sum $f+g$ continuous on?
2. Determine whether the function

$$f(x) = \begin{cases} \sin(x-\pi) & \text{if} \quad x < -\frac{\pi}{2} \\ 2x+\pi & \text{if} \quad -\frac{\pi}{2} \leq x \leq 3\pi \\ -\frac{7x}{2} + \frac{35\pi}{2} & \text{if} \quad x > 3\pi \end{cases}$$

is continuous on its domain.

3. Sketch the graph of a function which is continuous on the set $(-\infty,-1)\cup(-1,3]\cup(3,6]$ and it is discontinuous at 3.

4. Explain why the function

$$f(x)=[x^2-\sin^3(2x)][3x^2-\cos(x-3)] \quad \text{is continuous everywhere.}$$

5. Explain why the function

$$\sin\left(\frac{1}{x}\right) \quad \text{is not continuous at zero.}$$

6. Sketch the graph of one and only one function f which satisfies all the conditions listed below.

   a. f is defined and discontinuous at zero.
   b. $\lim_{x\to 0} f(x)=1$.
   c. $f(x)>0$ for all $x>0$.
   d. f is continuous on the interval $(0,\infty)$.

Simplifying Functions in the Evaluation of Limits

Compare the graph of the identity function $I(x)=x$ with the graph of the function

$$g(x)=\frac{x^2}{x} \quad \text{shown in Figure 8.5.}$$

The function g is not defined at zero and

$$g(x)=I(x) \quad \text{for every nonzero } x.$$

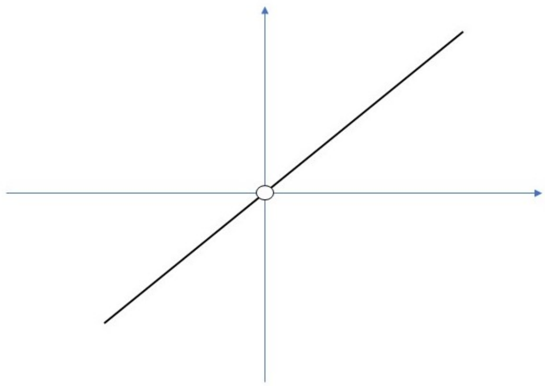


Figure 8.5: Graph of the function $g(x) = \dfrac{x^2}{x}$

Based on their graphs, we conclude that I and g are not equal. If they were, their graphs would be the same. Hence, the domains of two functions are important to determine whether they are equal.

Definition 8.3. Two functions $f(x)$ and $g(x)$ are equal if they have the *same* domain D and

$$f(x) = g(x) \quad \text{for every } x \text{ in } D.$$

The functions I and g are not equal because they do not have the same domain. The domain of I is $\mathbb{R}$ and the domain of g is $\mathbb{R} - \{0\}$. However,

$$I(x) = g(x) \quad \text{for } x \to 0$$

and their limits at zero are equal

$$\lim_{x\to 0} I(x) = 0 = \lim_{x\to 0} g(x).$$

In general, we have the following theorem, proved as Theorem 6.13 on page 161.

Theorem 8.4. *a. If $f(x) = g(x)$ for $x \to a^-$, then*

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} g(x).$$

b. If $f(x) = g(x)$ for $x \to a^+$, then

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x).$$

c. If $f(x) = g(x)$ for $x \to a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Theorem 8.4 justifies the simplification of functions to evaluate limits.

When we simplify a function f, we aim to obtain another function g which

- is continuous from the left at a number a and satisfies condition (a) of Theorem 8.4.
- is continuous from the right at a number a and satisfies condition (b) of Theorem 8.4.
- is continuous at a number a and satisfies condition (c) of Theorem 8.4.

To simplify a function we may

A. manipulate the function algebraically.

B. apply identities.

C. apply properties of the function.

We illustrate this in the next examples.

Example 8.4. The rational function

$$f(x) = \frac{x-1}{x^2-1} \quad \text{is simplified by factoring.}$$

Hence,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

This equality holds for all numbers except 1, and the everywhere continuous function $g(x) = x+1$ satisfies condition (c) of Theorem 8.4. Hence,

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \lim_{x\to 1} x+1 = g(1) = 2.$$

$\square$

Students must understand that the functions f and g are not equal. But their limits at 1 are equal.

Example 8.5. The function

$$f(x) = \frac{1-\cos^2 x}{\sin x} \quad \text{is not continuous at zero.}$$

To evaluate its limit at zero we apply the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Hence,

$$\frac{1-\cos^2 x}{\sin x} = \frac{\sin^2 x}{\sin x} = \sin x \qquad \text{for } x \to 0.$$

The condition (c) of Theorem 8.4 holds and

$$\lim_{x\to 0} \frac{1-\cos^2 x}{\sin x} = \lim_{x\to 0} \sin x = \sin 0 = 0.$$

$\square$

Example 8.6. The function

$$\ln(x^2-1) - \ln(x-1) \quad \text{is not continuous at 1.}$$

To evaluate the limit

$$\lim_{x\to 1} \ln(x^2-1) - \ln(x-1),$$

we apply the properties of the natural logarithmic function and the factorization of the function in Example 8.4. Thus,

$$\ln(x^2-1)-\ln(x-1)=\ln\left(\frac{x^2-1}{x-1}\right)=\ln(x+1) \quad \text{for } x\to 1.$$

The limit is

$$\lim_{x\to 1}\ln(x^2-1)-\ln(x-1)=\lim_{x\to 1}\ln(x+1)=\ln(2).$$

because the function $\ln(x+1)$ is continuous at 1. □

Objectives

Students should be able to

1. determine whether two given functions are equal.
2. simplify functions in order to evaluate limits.

Each of the questions below tests the objectives of this section. Review them and make up your own questions.

Exercises

7. For which x does the equality below hold?

$$\frac{x^3+2x^2-4x-8}{x^2-4}=x+2.$$

8. Is the function

$$f(x)=\frac{x^3+2x^2-4x-8}{x^2-4}$$

equal to the function $g(x)=x+2$?

9. Sketch the graphs of the functions f and g in question 2.

10. Evaluate the given limits by simplifying the given function.

    a. $\lim_{x \to 1} \dfrac{1 - \sec^2(x-1)}{x-1}$

    b. $\lim_{xto1} \ln(x-1) + \ln(x+1) - \ln(2x^2 - 2)$

    c. $\lim_{x \to 3} \dfrac{x^3 - 3x^2 + 2x - 6}{x-3}$

Finite Limits from the Right and Left

Students must be able to determine if the limit L of a function is from the right L^+ or left L^-.

In Figure 8.6 we have the limits of a function f from the right and left at a number a.

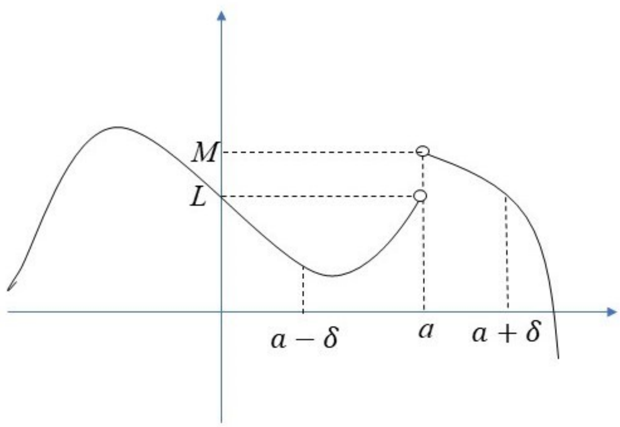


Figure 8.6: Limit at a number a from the left and right

It is clear that

$$\lim_{x \to a^-} f(x) = L \quad \text{and} \quad \lim_{x \to a^+} f(x)f(x) = M.$$

Moreover, for a positive number $\delta > 0$ we have that

$$f(x) < L \quad \text{for all } a - \delta < x < a$$

and

$$f(x) < M \quad \text{for all } a < x < a + \delta.$$

These indicate that on the sets $x \to a^-$ and $x \to a^+$, the values $f(x)$ "approach" L and M, respectively, both from the left. We write,

$$\lim_{x \to a^-} f(x) = L^- \quad \text{and} \quad \lim_{x \to a^+} f(x) = M^-.$$

In general, we define the limits at a number from left and right as follows.

Definition 8.5. a. The limit at a number a from the right of a function $f(x)$ is L from the right if

i. $\lim_{x \to a^+} f(x) = L$, and

ii. $f(x) > L$ for $x \to a^+$; that is, there is a positive number $\delta > 0$, such that

$$f(x) > L \quad \text{for any } 0 < x - a < \delta.$$

We write

$$\lim_{x \to a^+} f(x) = L^+.$$

b. The limit at a number a from the left of a function $f(x)$ is L from the right if

i. $\lim_{x \to a^-} f(x) = L$, and

ii. $f(x) > L$ for $x \to a^-$; that is, there is a positive number $\delta > 0$, such that

$$f(x) > L \quad \text{for any } 0 < a - x < \delta.$$

We write

$$\lim_{x \to a^-} f(x) = L^+.$$

c. The limit at a number a of a function $f(x)$ is L from the right if

i. $\lim_{x \to a} f(x) = L$, and

ii. $f(x) > L$ for $x \to a$; that is, there is a positive number $\delta > 0$, such that

$$f(x) > L \quad \text{for any } 0 < |x - a| < \delta.$$

We write

$$\lim_{x \to a} f(x) = L^{+}.$$

d. The limit at a number a from the right of a function $f(x)$ is L from the left if

i. $\lim_{x \to a^{+}} f(x) = L$, and

ii. $f(x) < L$ for $x \to a^{+}$; that is, there is a positive number $\delta > 0$, such that

$$f(x) < L \quad \text{for any } 0 < x - a < \delta.$$

We write

$$\lim_{x \to a^{+}} f(x) = L^{-}.$$

e. The limit at a number a from the left of a function $f(x)$ is L from the left if

i. $\lim_{x \to a^{-}} f(x) = L$, and

ii. $f(x) < L$ for $x \to a^{-}$; that is, there is a positive number $\delta > 0$, such that

$$f(x) < L \quad \text{for any } 0 < a - x < \delta.$$

We write

$$\lim_{x \to a^{-}} f(x) = L^{-}.$$

f. The limit at a number a of a function $f(x)$ is L from the left if

i. $\lim_{x\to a} f(x) = L$, and

ii. $f(x) < L$ for $x \to a$; that is, there is a positive number $\delta > 0$, such that

$$f(x) < L \quad \text{for any } 0 < |x-a| < \delta.$$

We write

$$\lim_{x\to a} f(x) = L^{-}.$$

Objectives

Students should be able to

1. determine visually whether the limit of a function is from the right or left.
2. determine whether the limit of a function is from the right or left.

Each of the questions below tests the objectives of this section. Review them and make up your own questions.

Exercises

11. Sketch the graph of one function whose limit at the number a is L from the left.
12. Sketch the graph of one function whose limit at the number a from the right is L from the right.

13. Evaluate the limits listed below and determine whether the limits are from the right or left.

    a. $\lim_{x \to 1} \frac{x+2}{x^2 + 4x + 4}$.

    b. $\lim_{x \to 2^-} x^3 + 3x^2 - 2x - 6$.

    c. $\lim_{x \to \pi/2^+} x \sin(2x)$

Chapter 9
Infinite Limits: Vertical Asymptotes

We recommend to teach infinite limits through the study of reciprocal functions.

The reciprocal function F of a function f with domain D_f is defined as

$$F(x) = \frac{1}{f(x)} \quad \text{for } x \in D_f \text{ so that } f(x) \neq 0.$$

Therefore, the function

$$f(x) = \frac{1}{F(x)} \quad \text{is the reciprocal of the function } F(x).$$

See Definition 4.12 (page 92).

The continuity of the reciprocal function F depends on the continuity of the function f.

Theorem 9.1. *If f is continuous at a number a and $f(a) \neq 0$, then its reciprocal*

$$F(x) = \frac{1}{f(x)} \quad \textit{is continuous at } a.$$

And,

$$\lim_{x \to a} F(x) = \frac{1}{f(a)}.$$

Example 9.1. The polynomial function $P(x) = x^2 + 1$ is positive for all x and continuous everywhere. Hence, its reciprocal

$$Q(x) = \frac{1}{x^2 + 1} \qquad \text{is continuous everywhere.}$$

□

Example 9.2. The polynomial function $P(x) = x^2 - 1$ is continuous everywhere. Since, $f(x) \neq 0$ for all x except 1 and -1, its reciprocal

$$Q(x) = \frac{1}{x^2 - 1} \qquad \text{is continuous on the set } (-\infty, -1) \cup (-1, 1) \cup (1, \infty).$$

$\square$

The evaluation of limits

$$\lim_{x \to a} \frac{1}{f(x)} \qquad \text{where } f(a) = 0$$

starts with the understanding of "large" and "small" numbers. At this moment, the quantitative value of a large and small number is not important. What it is important is the understanding of the *location* of small and large numbers on the real line.

On the real line all numbers on the right of zero are positives and all numbers on the left of zero are negatives.

In Figure 9.1 the number k is positive and the number s is negative. The quantity $k - s$ is the distance from s to k.

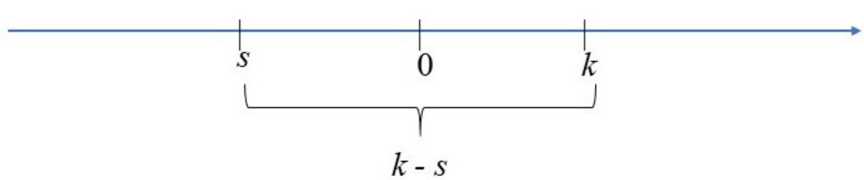


Figure 9.1: Location of the numbers s and k on the real line

A number s is smaller than a number k if and only if on the real line, the number s is on the left of the number k. We write $s < k$. Also, a number k is larger than a number s if and only if on the real line, the number k is on the right of the number s. We write $k > s$.

In Figure 9.1 the number s is smaller than the number k, and the number k is larger than the number s.

On the real line, all numbers are order related ($<$).

Definition 9.2. Two numbers k and s are order related,

$$s < k \quad \text{if and only if} \quad k - s > 0.$$

Observe that if $k - s > 0$, then $s - k < 0$. Then, two numbers k and s are order related,

$$k > s \quad \text{if and only if} \quad s - k < 0.$$

Looking at the real line, we have that

1. the farther a positive number is from zero, the larger it is. That is, a number is "very large" if it is positive and very far from zero.
2. the farther a negative number is from zero, the smaller it is. That is, a number is "very small" if it is negative and very far from zero.

Students must understand that on the real line, very large numbers are positive and very small numbers are negative. Numbers around zero are not very small.

Let us consider, next, the relationship between a number and its reciprocal.

Definition 9.3. The reciprocal of a non-zero number k is $\dfrac{1}{k}$.

Remark 9.4. *1. A non-zero number has only one reciprocal.*

2. The number zero is the only one without a reciprocal.

3. The only numbers which are equal to their reciprocals are -1 *and 1.*

4. The reciprocal of a non-zero rational number $\dfrac{p}{q}$ *is* $\dfrac{1}{\frac{p}{q}} = \dfrac{q}{p}$.

5. *If a positive number k is bigger than 1, then its reciprocal is positive and smaller than 1. That is,*

$$1 < k \quad \Rightarrow \quad 0 < \frac{1}{k} < 1 \quad \textit{as shown in Figure 9.2.}$$

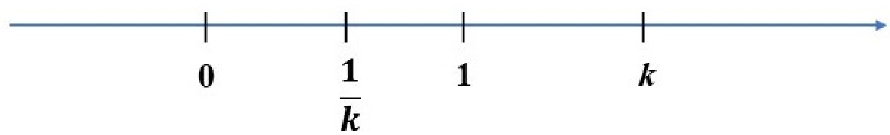


Figure 9.2: Location on the real line of the numbers $k > 1$ and its reciprocal $1/k$

6. *If a positive number k is smaller than 1, then its reciprocal is positive and larger than 1. That is,*

$$0 < k < 1 \quad \Rightarrow \quad \frac{1}{k} > 1 \qquad \textit{as shown in Figure 9.3.}$$

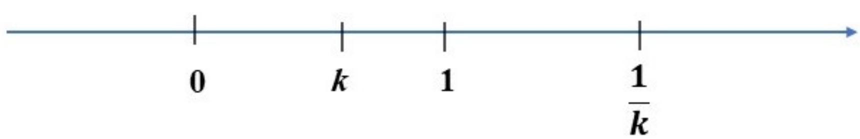


Figure 9.3: Location of the numbers $k < 1$ and its reciprocal $1/k$ on the real line

7. *If a negative number s is smaller than -1, then its reciprocal is negative and larger than -1. That is,*

$$s < -1 \quad \Rightarrow \quad -1 < \frac{1}{s} < 0 \qquad \textit{as shown in Figure 9.4.}$$

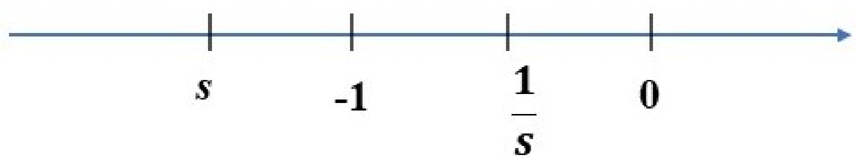


Figure 9.4: Location of the numbers $s < -1$ and its reciprocal $1/s$ on the real line

8. *If a negative number s is bigger than -1, then its reciprocal is negative and smaller than -1. That is,*

$$-1 < s < 0 \quad \Rightarrow \quad \frac{1}{s} < -1 \quad \textit{as shown in Figure 9.5.}$$

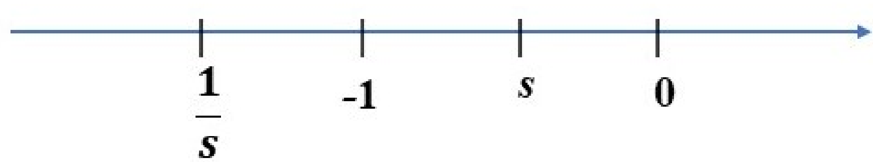


Figure 9.5: Location of the numbers $s > -1$ and its reciprocal $1/s$ on the real line

Figures 9.6 to 9.12 show the relationships between positive numbers and their reciprocals.

Figure 9.6 shows that

$$0 < k < m < 1,^{1} \quad \Rightarrow \quad 1 < \frac{1}{m} < \frac{1}{k}.$$

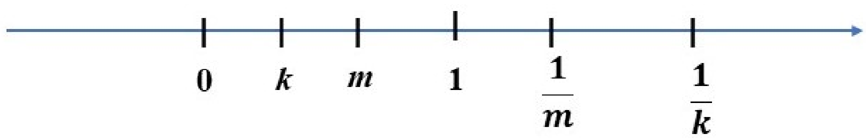


Figure 9.6: Order relation between the reciprocals of positive numbers close to zero

Conclusion 9.5. *The closer a positive number is to zero, the farther its reciprocal is from zero.*

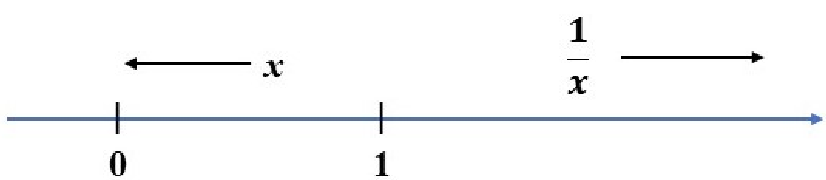


Figure 9.7: Diagram of Conclusion 9.5

Figure 9.8 shows that

$$1 < k < m \quad \Rightarrow \quad 0 < \frac{1}{m} < \frac{1}{k} < 1.$$

Conclusion 9.6. *The farther is a positive from zero, the closer its reciprocal is to zero.*

Figure 9.10 shows that

$$-1 < m < s < 0 \quad \Rightarrow \quad \frac{1}{s} < \frac{1}{m} < -1.$$

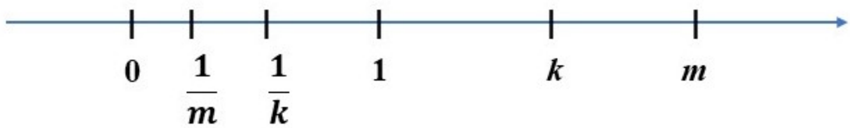


Figure 9.8: Order relation between the reciprocals of numbers greater than 1

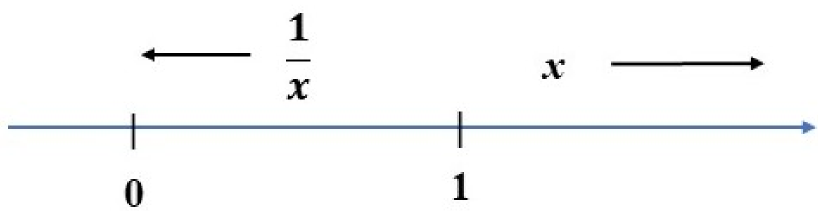


Figure 9.9: Diagram of Conclusion 9.6

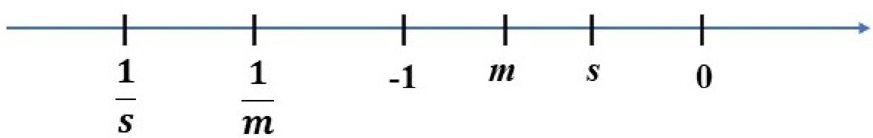


Figure 9.10: Order relation between the reciprocals of negative numbers close to zero

Conclusion 9.7. *The closer a negative number is to zero, the farther its reciprocal is from zero.*

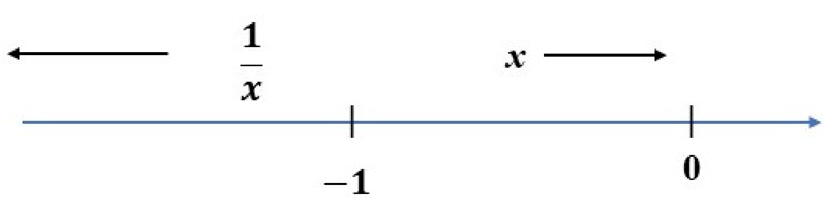


Figure 9.11: Diagram of Conclusion 9.7

Figure 9.12 shows that

$$m < s < -1 \quad \Rightarrow -1 < \frac{1}{s} < \frac{1}{m} < 0.$$

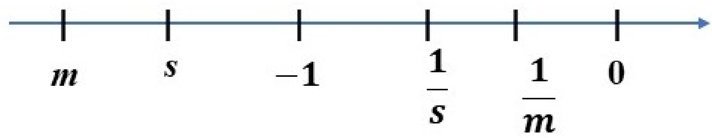


Figure 9.12: Order relation between the reciprocals of small negative numbers

Conclusion 9.8. *The farther is a negative number from zero, the closer its reciprocal is to zero.*

Figure 9.13: Diagram of Conclusion 9.8

Students must fully understand Conclusions 9.5 to 9.8 (pages 204 to 206) and be able to describe the relationship between a function and its reciprocal.

Objectives

Students should be able to

1. Identify the set where the reciprocal function $\dfrac{1}{f(x)}$ of a function $f(x)$ is continuous.
2. Establish the order relation between any two numbers.
3. Establish the order relation between any two reciprocal numbers.

Each of the questions below tests the objectives of this section. Identify the objective(s) tested by each exercise. Make up your own exercises.

Exercises

1. Give the set where the function

$$\frac{1}{2x^2 - x - 6} \quad \text{is continuous.}$$

2. Give an example of a function $f(x)$ such that its reciprocal

$$\frac{1}{\sqrt{f(x)}} \quad \text{is continuous everywhere.}$$

3. Explain why the function

$$\frac{1}{x^3 - 3} \quad \text{is not continuous everywhere.}$$

4. Give two positive numbers a and b such that

$$a < 10^{12} < b.$$

5. Give two negative numbers a and b such that

$$\frac{1}{a} < -10^{-12} < \frac{1}{b}.$$

6. Give the reciprocal of the numbers listed below.

   a. $\frac{3}{8}$,
   b. -5,
   c. $\frac{5-7}{\sqrt{2}+2}$,

d. 3^{-6}.

7. Establish the order relation between any two of the numbers given below.

$$-\sqrt{65}, 1-\sqrt{3}, -1, \pi, \frac{1}{\sqrt{2\pi}}, 3.$$

8. Give a number n such that

$$\frac{3}{5} < n < 1.$$

9. Give an integer n such that

$$\frac{8}{15} < \frac{6}{n}.$$

10. Give all positive integers n such that

$$\frac{n}{6} < \frac{7}{n} < \frac{5n}{9}.$$

Complete solutions are provided on page **??**.

The Symbols ∞ and $-\infty$

The symbols infinity (∞) and negative infinity ($-\infty$) are used to represent unbounded sets of numbers so that under the order relation ($<$) they increase or decrease *without bound*.

On the real line, unbounded sets have no boundaries above or below. Thus, their elements are larger or smaller than any given numbers, whatever the values of these given numbers may be.

Definition 9.9. A subset S of $\mathbb{R}$ is *unbounded above* if for any $V > 0$, there is an $x \in S$ such that $x > V$.

Note that if $V > 0$ is a very large number and $x > V$, then the number x is also a very large number. That is, all numbers on the right of V are "infinitely" large.

Example 9.3. The set

$$\{2, 4, 6, 8, 10, 12...\} = \{2n|\ n \in \mathbb{Z}^+\} \quad \text{of even positive integers}$$

is unbounded above, because for any positive number $V > 0$, there is an integer $n > V$, and the even integer $2n$ is bigger than n and therefore bigger than V. $\square$

The expression $x \to \infty$ reads "x *tends* to infinity" and it indicates a set of infinitely large numbers. That is, for *any* large V (it does not matter how large) there is always a number larger than V. This is what we mean by "numbers which increase without bound."

It is incorrect to read $x \to \infty$ as "x approaches infinity."

On the real line, there is no place for infinity (∞), so numbers cannot approach something which is nowhere.

Definition 9.10. A subset S of $\mathbb{R}$ is *unbounded below* if for any $U < 0$, there is an $x \in S$ such that $x < U$.

Note that if $U < 0$ is a very small number and if $x < U$, then the number x is also a very small number. That is, all numbers on the left of U are "infinitely" small.

Example 9.4. The set

$$\{-2, -4, -6, -8, -10, -12...\} = \{2n|\ n \in \mathbb{Z}^-\} \quad \text{of even negative integers}$$

is unbounded below, because for any negative number $U < 0$, there is an integer $m < U$, and the even integer $2m$ is smaller than m and therefore smaller than U. $\square$

The expression $x \to -\infty$ reads "the number x *tends* to negative infinity" and it indicates a set of infinitely small numbers. That is, for *any* small U (it does not matter how small) there is always an x smaller than U. This is what we mean by "numbers which decrease without bound."

It is incorrect to read $x \to -\infty$ as "x approaches negative infinity."

On the real line, there is no place for negative infinity ($-\infty$), so numbers x cannot approach something which is nowhere.

Study carefully, the steps we follow to sketch the graph of the reciprocal function

$\dfrac{1}{I(x)} = \dfrac{1}{x}$ of the everywhere continuous identity function $I(x) = x$.

1. **Identify the set where the function is positive.**
   The function

   $I(x)$ is positive and continuous on $(0, \infty)$.

   Hence,

   $\dfrac{1}{I(x)}$ is also positive and continuous on $(0, \infty)$.

2. **Identify the set where the function is negative.**
   The function

   $I(x)$ is negative and continuous on $(-\infty, 0)$.

   Hence,

   $\dfrac{1}{I(x)}$ is also negative and continuous on $(-\infty, 0)$.

3. **Identify the set where the function is zero.**
   Since $I(0) = 0$, the function

   $\dfrac{1}{I(x)}$ is undefined at zero.

4. **Identify the set where the function is positive and very large.** From the graph of the identity function, we see that the value of

$$I(a) = a \quad \text{is very far from zero for a very large } a > 1.$$

Then, by (5) of Remark 9.4 (page 201), the reciprocal

$$\frac{1}{I(a)} \quad \text{is positive and very close to zero.}$$

Also, by Figure 9.8

$$\text{if } 1 < a < b, \text{ then } 0 < \frac{1}{I(b)} = \frac{1}{b} < \frac{1}{a} = \frac{1}{I(a)} < 1.$$

Conclusion 9.11. *The farther is x from 1, the closer is the positive number $\dfrac{1}{I(x)}$ to zero.*

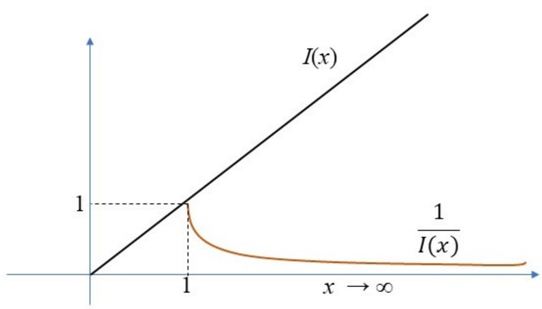


Figure 9.14: Diagram of Conclusion 9.11

5. **Identify the set where the function is positive and very close to zero.** The function $I(x)$ is positive and very close to zero for $x \to 0^+$. Then, by (6) of Remark 9.4 (page 201), the reciprocal

$$\frac{1}{I(x)} \quad \text{is positive and far from zero for } x \to 0^+.$$

Also, by Figure 9.6

$$\text{if } 0 < a < b < 1, \text{ then } 1 < \frac{1}{I(b)} = \frac{1}{b} < \frac{1}{a} = \frac{1}{I(a)}.$$

Conclusion 9.12. *The closer is x to zero, the farther is $\dfrac{1}{I(x)}$ from 1.*

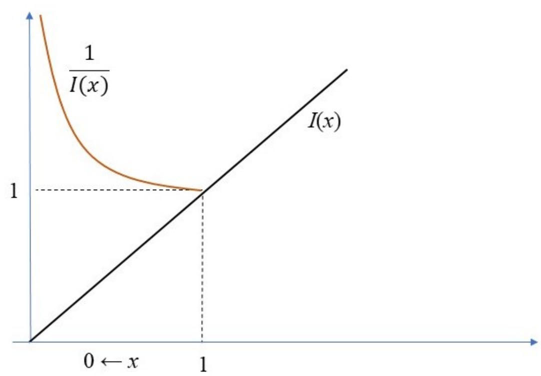


Figure 9.15: Diagram of Conclusion 9.12

6. **Identify the set where the function is negative and very small.**
   The function $I(x) = x$ is negative and very far from -1 for $a < -1$. Then, by (7) of Remark 9.4 (page 201), the reciprocal

   $$\frac{1}{I(x)} \quad \text{is negative and very close to zero for } a < -1.$$

   Also, by Figure 9.12

   $$\text{if } a < b < -1, \text{ then } -1 < \frac{1}{I(b)} = \frac{1}{b} < \frac{1}{a} = \frac{1}{I(a)} < 0.$$

   Conclusion 9.13. *The farther is the negative number x from zero, the closer is the negative number $\dfrac{1}{I(x)}$ to zero.*

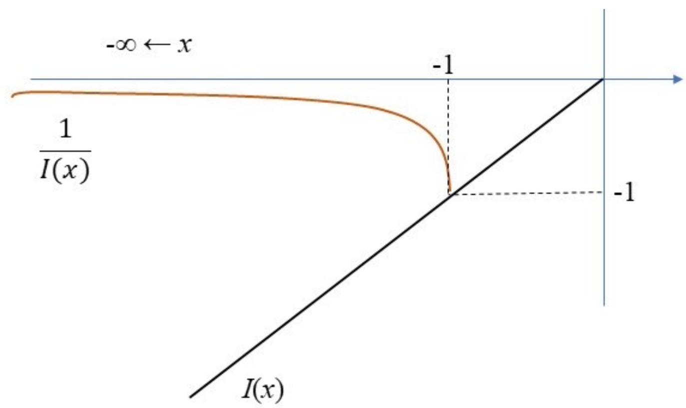


Figure 9.16: Diagram of Conclusion 9.13

7. **Identify the set where the function is negative and close to zero.** The function $I(x)$ is negative and very close to zero for $x \to 0^-$. Then, by (8) of Remark 9.4 (page 201), the reciprocal

$$\frac{1}{I(x)} \quad \text{is negative and very far from zero for } x \to 0^-.$$

Also, by Figure 9.8, if

$-1 < a < b < 0$, then $-1 < \dfrac{1}{I(b)} = \dfrac{1}{b} < \dfrac{1}{a} = \dfrac{1}{I(a)} < 0.$

Conclusion 9.14. *The closer is the negative number x to zero, the farther is the negative number $\dfrac{1}{I(x)}$ from zero.*

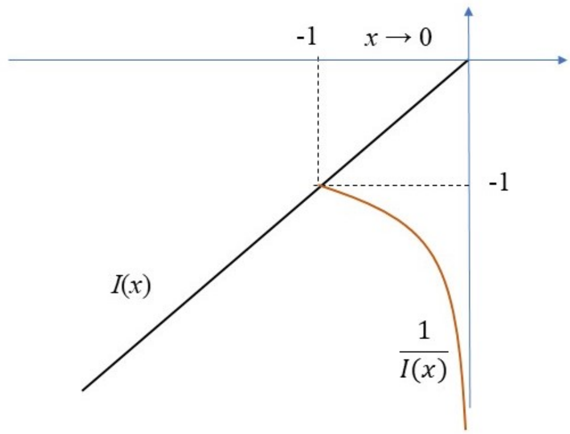


Figure 9.17: Diagram of Conclusion 9.14

8. **Sketching the reciprocal function.**
   We sketch the graph of $\dfrac{1}{I(x)}$, by putting together the information we obtained from Conclusions 9.11 to 9.14 (from pages 211 to 213).

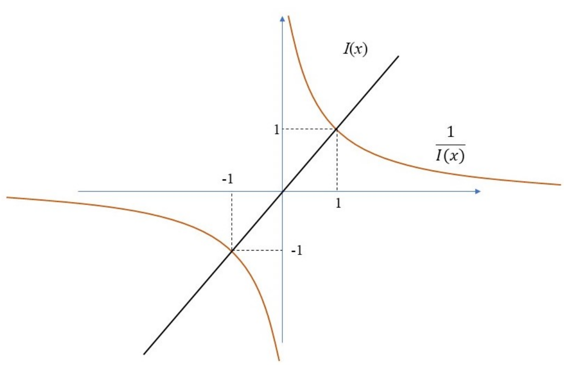


Figure 9.18: The graph of the reciprocal function $\dfrac{1}{I(x)} = \dfrac{1}{x}$

Similarly, we sketch the graph of the reciprocal of the piecewise function.

$$f(x) = \begin{cases} x & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

The domain of this function is $\mathbb{R}$, and $f(x) \neq 0$ for any x.

Figure 9.19 shows the graph of the function f.

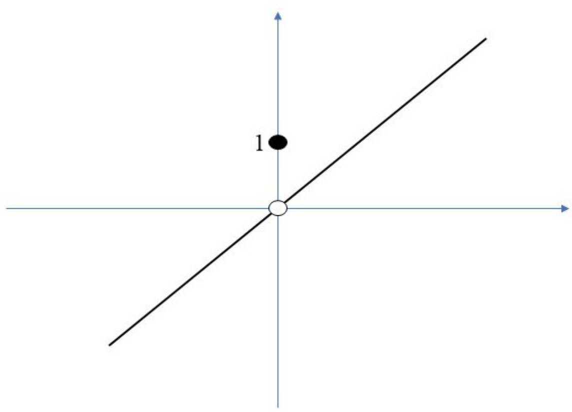


Figure 9.19: Graph of the piecewise function $f(x)$

Its reciprocal function is

$$\frac{1}{f(x)} = \begin{cases} \frac{1}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

Hence,

$$\frac{1}{f(x)} = \frac{1}{I(x)} \quad \text{for any nonzero } x.$$

On the other hand,

$$\frac{1}{f(0)} = 1.$$

Applying the graph of the function

$$\frac{1}{I(x)} = \frac{1}{x} \quad \text{for } x \neq 0,$$

we obtain the graph of $\dfrac{1}{f(x)}$.

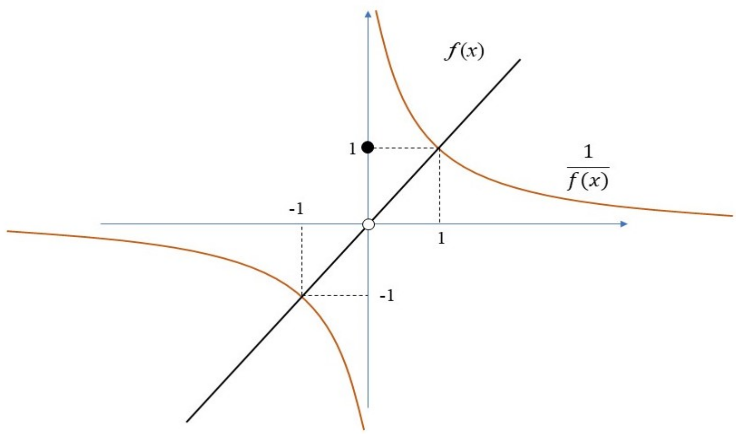


Figure 9.20: Graph of the function $\dfrac{1}{f(x)}$

Comparing the graphs of the functions $f(x)$ and $I(x)$, we see that

$$I(0) = 0 \qquad \text{and} \qquad \lim_{x\to 0} I(x) = 0$$

and

$$f(0) = 1 \qquad \text{and} \qquad \lim_{x\to 0} f(x) = 0.$$

The graphs of the reciprocal functions

$$\frac{1}{I(x)} \quad \text{and} \quad \frac{1}{f(x)} \quad \text{are equal around zero}$$

See Figure 9.18 on page 214 and Figure 9.20 on page 216.

This is due to the fact that

$$I(x) = f(x) \quad \text{for } x \to 0.$$

Also, the set

$$\left\{ \frac{1}{I(x)} \middle| x \to 0^+ \right\} \quad \text{is unbounded.}$$

Indeed, in Figure 9.21 below,

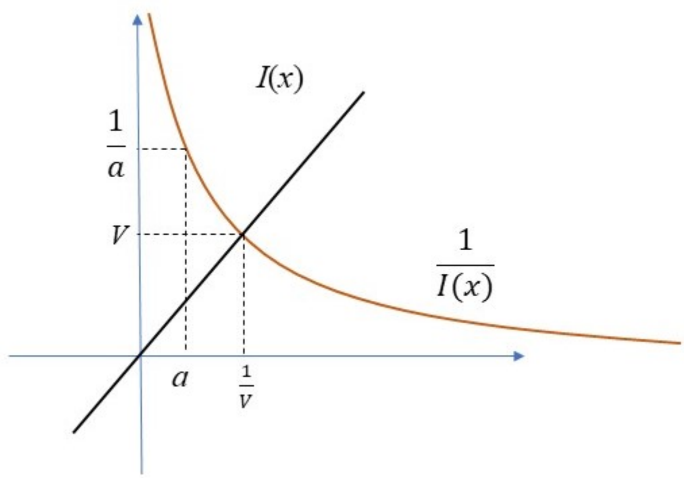


Figure 9.21: Graph of the function $\dfrac{1}{I(x)}$ for $x \to 0^+$

we see that for any $V > 0$, there is a number a close to zero from the right, such that

$$\frac{1}{I(a)} = \frac{1}{a} > V.$$

The same is true for the set

$$\left\{ \frac{1}{I(x)} \middle| x \to 0^- \right\}.$$

These two examples take us to the study of vertical and horizontal asymptotes.

Objectives

Students should be able to

1. Give examples of unbounded sets.
2. Sketch the graph of reciprocal functions.

Each of the questions below tests the objectives of this section. Identify the objective(s) tested by each exercise. Make up your own exercises.

Exercises

11. Apply Definition 9.9 to show that the set

$$S = \{e^{2x/3} \mid x \in \mathbb{R}\}$$

is unbounded above.

12. Give the definition of a bounded set.
Hint. Consider the negations of Definition 9.9 and Definition 9.10.

13. Follow the steps 1-8 above on page 210, to sketch the graph of the reciprocal of the basic square function $S(x) = x^2$.

14. Give the sketch of the graph of the function $\dfrac{1}{f(x)}$ where

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \neq 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$$

15. Explain why the set

$$\left\{ \frac{1}{I(x)} \,\middle|\, x \to 0^- \right\}$$

is unbounded below and bounded above.

Complete solutions are provided on page 345.

Vertical Asymptotes of Reciprocal Functions

We start our study of vertical asymptotes with the definitions of unbounded functions. See the negation of Definition 4.2 on page 4.

Definition 9.15. A function f is

a. *unbounded above* on a set I, if for any number V, there is a number $x \in I$, such that $f(x) > V$.

b. *unbounded below* on a set I, if for any number N, there is a number $x \in I$, such that $f(x) < N$.

c. *unbounded* on a set I, if it is unbounded above and below. That is, if for any number B, there are numbers a, b in I, such that $f(a) > B$ and $f(b) < B$.

The definition of bounded functions is the negation of Definition 9.15.

Convince yourself of the veracity of the statements listed below.

a. The identity function $I(x)$ and its reciprocal $\dfrac{1}{I(x)}$ are unbounded on their domains.

b. The square function $S(x) = x^2$ is unbounded above, and bounded below.

c. The sine and cosine functions are bounded.

If a function $f(x)$ is unbounded above on a set S, then we say that $f(x)$ *tends* to infinity on S. We write,

$$f(x) \to \infty \quad \text{on the set } S.$$

If a function $f(x)$ is unbounded below, then we say that $f(x)$ *tends* to negative infinity. We write,

$$f(x) \to -\infty \quad \text{on the set } S.$$

For the same reasons indicated earlier, it is incorrect to say that $f(x)$ approaches infinity or negative infinity.

By Definition 8.5, the notation

$$\lim_{x \uparrow K} f(x) = 0^+ \quad \text{indicates two facts}$$

- $\lim_{x \uparrow K} f(x) = 0$, and
- $f(x) > 0$ for $x \to K^+, K^-, K$.

Similarly,

$$\lim_{x \uparrow K} f(x) = 0^- \quad \text{indicates two facts}$$

- $\lim_{x \uparrow K} f(x) = 0$, and
- $f(x) < 0$ for $x \to K^+, K^-, K$.

We re-write Conclusions 9.11 to 9.14 (pages 211 to 213).

Conclusion 9.11. The farther is the number x from zero, the closer is the positive number $\dfrac{1}{I(x)}$ to zero.

$x \to \infty$	the farther is the number x from zero
$\dfrac{1}{I(x)} \to 0^+$	the closer is the positive number $\dfrac{1}{I(x)}$ to zero.

Conclusion 9.12. The closer is the positive number x to zero, the farther is the positive number $\frac{1}{I(x)}$ from zero.

$x \to 0^+$	the closer is the positive number x to zero
$\frac{1}{I(x)} \to \infty$	the farther is the positive number $\frac{1}{I(x)}$ from zero.

Conclusion 9.13. The closer is the negative number x to zero, the farther the negative number $\frac{1}{I(x)}$ is from zero.

$x \to -\infty$	the farther is the negative number x from zero
$\frac{1}{I(x)} \to 0^-$	the closer is the negative number $\frac{1}{I(x)}$ to zero.

Conclusion 9.14. The closer is the negative number x to zero, the farther is the negative number $\frac{1}{I(x)}$ from zero.

$x \to 0^-$	closer is the negative number x to zero
$\frac{1}{I(x)} \to -\infty$	the farther is the number $\frac{1}{I(x)} < 0$ from zero.

Graphically, if the values $f(x)$ of a function increase without bound as x tends to a number K from the right or left, then the vertical line $x = K$ is an asymptote of the function f. We write

$$\lim_{x \to K^+} f(x) = \infty \qquad \text{or} \qquad \lim_{x \to K^-} f(x) = \infty.$$

From Figure 9.15 (page 212), the line $x = 0$ is a vertical asymptote of the function $\frac{1}{I(x)}$, and

$$\lim_{x \to 0^+} \frac{1}{I(x)} = \infty.$$

Also, if the values $f(x)$ of a function decrease without bound as x tends to a number K from the right or left, then the vertical line $x = K$ is an vertical asymptote of the function f. We write

$$\lim_{x \to K^+} f(x) = -\infty \qquad \text{or} \qquad \lim_{x \to K^-} f(x) = -\infty.$$

From Figure 9.17 (page 214), the line $x = 0$ is a vertical asymptote of the function $\frac{1}{I(x)}$, and

$$\lim_{x \to 0^-} \frac{1}{I(x)} = -\infty.$$

The definition of vertical asymptotes is given in terms of limits, as in Definition 4.6 (page 83).

Definition 9.16. A vertical line $x = K$ is a vertical asymptote of the function $f(x)$, if any of the limits listed below holds.

a. $\lim_{x \to K^+} f(x) = \infty$,

b. $\lim_{x \to K^-} f(x) = \infty$,

c. $\lim_{x \to K^+} f(x) = -\infty$,

d. $\lim_{x \to K^-} f(x) = -\infty$.

If a function is not defined at a number a, then it may or may not have a vertical asymptote at a.

See that the reciprocal of the piecewise function, sketched in Figure 9.20 (page 216), is nonzero for all numbers. However, it has a vertical asymptote.

Students must evaluate limits to determine whether a function has a vertical asymptote or not. That is, they must apply Definition 9.16.

To apply Definition 9.16, we must determine the number K where the function may have a vertical asymptote.

For reciprocal functions, we apply Theorem 4.13 (page 94). That is, the reciprocal of a function f has a vertical asymptote at the number K if and only if either

$$\lim_{x \uparrow K} f(x) = 0^{+} \quad \text{or} \quad \lim_{x \uparrow K} f(x) = 0^{-}.$$

In summary, we have the following theorem for reciprocal functions.

Theorem 9.17. *a. If* $f(x) > 0$ *for* $x \to K^{+}$ *(*K^{-}*)(*K*) and*

$$\lim_{x \to K^{+}} f(x) = 0 \quad \left(\lim_{x \to K^{-}} f(x) = 0 \right) \quad \left(\lim_{x \to K} f(x) = 0 \right),$$

then

$$\lim \frac{1}{f(x)} = \infty.$$

b. If $f(x) < 0$ *for* $x \to K^{+}$ *(*K^{-}*)(*K*) and*

$$\lim_{x \to K^{+}} f(x) = 0 \quad \left(\lim_{x \to K^{-}} f(x) = 0 \right) \quad \left(\lim_{x \to K} f(x) = 0 \right),$$

then

$$\lim \frac{1}{f(x)} = -\infty.$$

In either case, the line $x = K$ is a vertical asymptote of the function $\dfrac{1}{f(x)}$.

In short, Theorem 9.17 is saying that

$$\text{if} \quad \lim_{x \uparrow K} f(x) = 0^+ \quad \text{then} \quad \lim_{x \uparrow K} \frac{1}{f(x)} = \infty.$$

and

$$\text{if} \quad \lim_{x \uparrow K} f(x) = 0^- \quad \text{then} \quad \lim_{x \uparrow K} \frac{1}{f(x)} = -\infty.$$

In the next examples we show the correct application of Theorem 9.17.

Example 9.5. To evaluate the limit

$$\lim_{x \to 0^+} \csc x$$

we consider the cosecant function as the reciprocal of the sine function. From the graph of sine

$$\lim_{x \to 0^+} \sin x = 0^+$$

and by Theorem 9.17

$$\lim_{x \to 0^+} \csc x = \infty.$$

□

Example 9.6. To evaluate the limit

$$\lim_{x \to 0} \frac{1}{x^n}$$

with n a positive integer, we consider the limits

$$\lim_{x \to 0^+} x^n \quad \text{and} \quad \lim_{x \to 0^-} x^n.$$

If n is even, then

$$\lim_{x \to 0} x^n = 0^+.$$

If n is odd, then

$$\lim_{x \to 0^+} x^n = 0^+ \quad \text{and} \quad \lim_{x \to 0^-} x^n = 0^-.$$

By Theorem 9.17

$$\lim_{x \to 0} \frac{1}{x^n} = \infty \quad \text{if } n \text{ is even.}$$

$$\lim_{x \to 0^+} \frac{1}{x^n} = \infty \quad \text{if } n \text{ is odd.}$$

$$\lim_{x \to 0^-} \frac{1}{x^n} = -\infty \quad \text{if } n \text{ is odd.}$$

$\square$

Example 9.7. To evaluate the limit

$$\lim_{x \to 1} \frac{1}{x^2 - 1}$$

we consider the limits

$$\lim_{x \to 1^+} \frac{1}{x^2 - 1} \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{x^2 - 1}.$$

Since

$$x^2 - 1 > 0 \quad \text{for } x \to 1^+ \quad \text{and} \quad x^2 - 1 < 0 \quad \text{for } x \to 1^-,$$

by Theorem 9.17

$$\lim_{x \to 1^+} \frac{1}{x^2 - 1} = \infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{x^2 - 1} = -\infty.$$

Therefore,

$$\lim_{x \to 1} \frac{1}{x^2 - 1} \quad \text{does not exist.}$$

$\square$

Students must avoid erroneous interpretations of Theorem 9.17.

To determine vertical asymptotes of reciprocal functions, we may follow the three steps related to Theorem 9.17 listed below.

1. Find numbers K such that $\lim_{x \uparrow K} f(x) = 0$.
2. Determine whether $f(x) > 0$ or $f(x) < 0$ for $x \to K^+, K^-, K$.
3. If the above two steps hold, then $x = K$ is a vertical asymptote of the function $\dfrac{1}{f(x)}$.

> The second step is necessary. Example 4.8 (92) shows that it is not true that if $\lim\limits_{x \uparrow K} f(x) = 0$, then the line $x = K$ is a vertical asymptote of the function $\dfrac{1}{f(x)}$.

Example 9.8. To determine the vertical asymptotes of the function

$$f(x) = \frac{1}{\ln x},$$

we find all numbers where the limit of the function $\ln x$ is zero. We have

$$\lim_{x \to 1} \ln x = 0$$

and the number 1 is the only such.

Since

$$\ln x > 0 \quad \text{for } x \to 1^+ \quad \text{and} \quad \ln x < 0 \quad \text{for } x \to 1^-,$$

we conclude by Theorem 9.17 that

$$\lim_{x \to 1^+} \frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{\ln x} = -\infty.$$

Thus, the vertical line $x = 1$ is a vertical asymptote of the function $f(x) = \dfrac{1}{\ln x}$. □

Example 9.9. The function

$$f(x) = (x-2)\ln x \quad \text{is continuous at 1 and 2.}$$

We have the limits,

$$\lim_{x\to 1}(x-2)\ln x = (-1)\ln 1 = 0 \quad \text{and} \quad \lim_{x\to 2}(x-2)\ln x = (0)\ln 2 = 0.$$

On the other hand,

a. $f(x) = (x-2)\ln x < 0$ for $x \to 1^+$,

b. $f(x) = (x-2)\ln x > 0$ for $x \to 1^-$,

c. $f(x) = (x-2)\ln x > 0$ for $x \to 2^+$,

d. $f(x) = (x-2)\ln x < 0$ for $x \to 2^-$.

By Theorem 9.17

a. $\displaystyle\lim_{x\to 1^+} \frac{1}{(x-2)\ln x} = -\infty$,

b. $\displaystyle\lim_{x\to 1^-} \frac{1}{(x-2)\ln x} = \infty$,

c. $\displaystyle\lim_{x\to 2^+} \frac{1}{(x-2)\ln x} = \infty$,

d. $\displaystyle\lim_{x\to 2^-} \frac{1}{(x-2)\ln x} = -\infty$.

The vertical lines $x = 1$ and $x = 2$ are vertical asymptotes of the function $\dfrac{1}{(x-2)\ln x}$. □

Students must be able to sketch the graph of a reciprocal function around a number K, of any given function whose limit at K is zero. Figure 9.22 and Figure 9.23 are examples of these graphic representations.

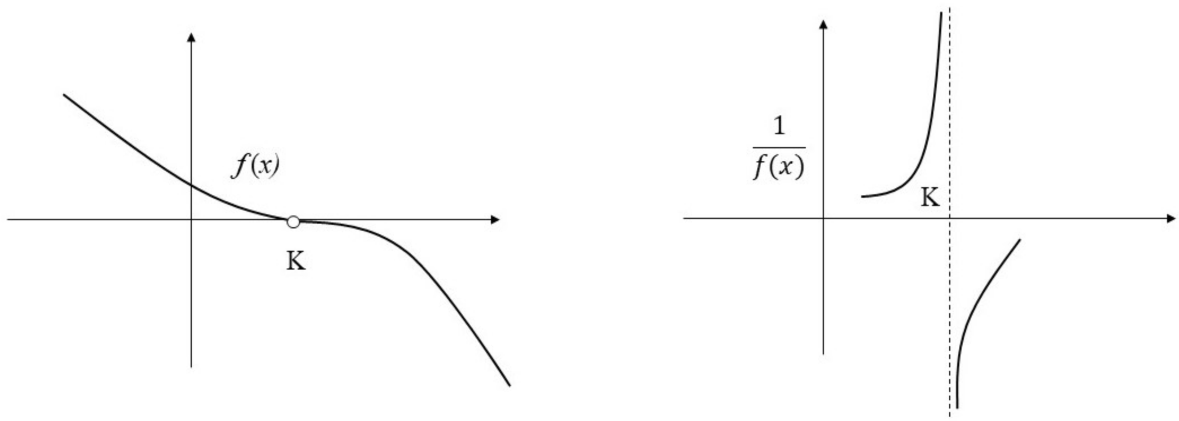


Figure 9.22: Graph of the reciprocal function $\dfrac{1}{f(x)}$ of the given function $f(x)$

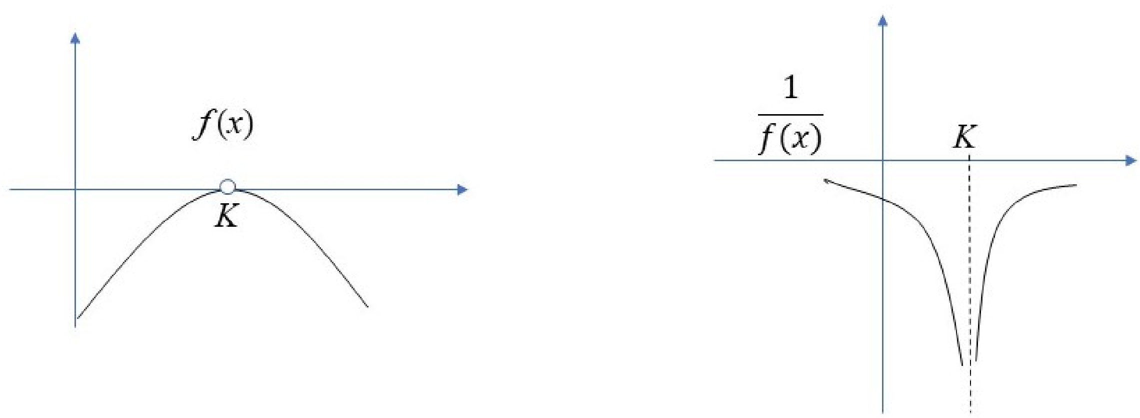


Figure 9.23: Graph of the reciprocal function $\dfrac{1}{f(x)}$ of the given function $f(x)$

Objectives

Students should be able to

1. determine whether a function is bounded or unbounded.
2. evaluate infinite limits of reciprocal functions.

3. find the vertical asymptotes of reciprocal functions.
4. sketch the graphs of reciprocal functions.

Each of the questions below tests the objectives of this section. Identify the objective(s) tested by each exercise. Make up your own exercises.

Exercises

16. Determine whether the functions listed below are unbounded (below or above) on the indicated range. Explain.

    a. $f(x) = x \sin x$ for all $x \in \mathbb{R}$

    b. $g(x) = \tan x$ for all $0 < x < \dfrac{\pi}{2}$

    c. $h(x) = \ln x$ for $0 < x < 1$

17. Give an example of a function which is bounded below and unbounded above on the interval $(0, 1)$.

18. Evaluate the limits listed below.

    a. $\displaystyle\lim_{x \to 2^+} \frac{1}{x^2 - 4}$

    b. $\displaystyle\lim_{x \to 2^-} \frac{1}{x^2 - 4}$

    c. $\displaystyle\lim_{x \to \pi^+} \sec x$

    d. $\displaystyle\lim_{x \to 2^+} \frac{1}{\ln(x - 1)}$

19. Find the vertical asymptotes of the functions listed below.

    a $f(x) = \dfrac{1}{x^3 - 8}$

b $g(x) = \dfrac{1}{\ln(x^2 - 1)}$

c $h(x) = \sec(x - \pi)$ on the interval $[0, 2\pi]$

20. Sketch the graph of the functions listed below.

a $f(x) = \ln(x - 1)$

b $g(x) = \dfrac{1}{\ln(x - 1)}$

21. Explain why the application of Theorem 9.17 (page 224), shown below is incorrect. Give its correct application.

$$\begin{aligned}
\lim_{x \to 1^+} \frac{1}{(x-1)\sin(\pi x)} &= \lim_{x \to 1^+} \frac{1}{(x-1)} \frac{1}{\sin(\pi x)} \\
&= \lim_{x \to 1^+} \frac{1}{(x-1)} \lim_{x \to 1^+} \frac{1}{\sin(\pi x)} \\
&= \lim_{x \to 1^+} \frac{1}{0^+} \lim_{x \to 1^+} \frac{1}{0^-} \\
&= (\infty)(-\infty) = -\infty.
\end{aligned}$$

Complete solutions are provided on page 348.

Vertical Asymptotes of Quotient Functions

To establish the vertical asymptotes of quotient functions

$$\frac{f(x)}{g(x)},$$

we must consider them as the product

$$\frac{f(x)}{g(x)} = f(x) \left(\frac{1}{g(x)} \right). \tag{9.1}$$

Definition 9.16 indicates that a quotient function has a vertical asymptote $x = K$ if any of the limits at K, K^+, or K^- is either infinite or negative infinite.

By parts (e) and (f) of Theorem 5.1 (page 101), the function (9.1) has a vertical asymptote at K if either

- $\lim\limits_{x \uparrow K} f(x) = \pm\infty$ and $\lim\limits_{x \uparrow K} \dfrac{1}{g(x)} = L \neq 0$, or
- $\lim\limits_{x \uparrow K} f(x) = L \neq 0$ and $\lim\limits_{x \uparrow K} \dfrac{1}{g(x)} = \pm\infty$.

By parts (g), (h), and (i) of Theorem 5.1 (page 101), the function (9.1) has a vertical asymptote at K if

$$\lim_{x \uparrow K} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \uparrow K} \frac{1}{g(x)} = \pm\infty.$$

These claims are stated in the following theorem.

Theorem 9.18. *A. If*

$$\lim_{x \uparrow K} f(x) = L \neq 0, \quad \lim_{x \uparrow K} \frac{1}{g(x)} = \infty \quad \textit{and} \quad \lim_{x \uparrow K} \frac{1}{G(x)} = -\infty,$$

then

a. $\lim\limits_{x \uparrow K} \dfrac{f(x)}{g(x)} = \infty$ *if* $L > 0$.

b. $\lim\limits_{x \uparrow K} \dfrac{f(x)}{g(x)} = -\infty$ *if* $L < 0$.

c. $\lim\limits_{x \uparrow K} \dfrac{f(x)}{G(x)} = \infty$ *if* $L < 0$.

d. $\lim\limits_{x \uparrow K} \dfrac{f(x)}{G(x)} = -\infty$ *if* $L > 0$.

B. If

$$\lim_{x\uparrow K} f(x) = \infty, \quad \lim_{x\uparrow K} \frac{1}{g(x)} = L \neq 0 \quad \textit{and} \quad \lim_{x\uparrow K} F(x) = -\infty,$$

then

a. $\lim_{x\uparrow K} \dfrac{f(x)}{g(x)} = \infty \quad$ *if* $L > 0$.

b. $\lim_{x\uparrow K} \dfrac{f(x)}{g(x)} = -\infty \quad$ *if* $L < 0$.

c. $\lim_{x\uparrow K} \dfrac{F(x)}{g(x)} = -\infty \quad$ *if* $L > 0$.

d. $\lim_{x\uparrow K} \dfrac{F(x)}{g(x)} = \infty \quad$ *if* $L < 0$.

C. If

$$\lim_{x\uparrow K} f(x) = \infty, \quad \textit{and} \quad \lim_{x\uparrow K} \frac{1}{g(x)} = \infty,$$

then

$$\lim_{x\uparrow K} \frac{f(x)}{g(x)} = \infty.$$

D. If

$$\lim_{x\uparrow K} f(x) = \infty, \quad \textit{and} \quad \lim_{x\uparrow K} \frac{1}{g(x)} = -\infty,$$

then

$$\lim_{x\uparrow K} \frac{f(x)}{g(x)} = -\infty.$$

E. If

$$\lim_{x\uparrow K} f(x) = \infty, \quad \textit{and} \quad \lim_{x\uparrow K} \frac{1}{g(x)} = -\infty,$$

then

$$\lim_{x\uparrow K} \frac{f(x)}{g(x)} = -\infty.$$

F. If

$$\lim_{x\uparrow K} f(x) = -\infty, \quad \text{and} \quad \lim_{x\uparrow K} \frac{1}{g(x)} = -\infty,$$

then

$$\lim_{x\uparrow K} \frac{f(x)}{g(x)} = \infty.$$

To conclude correctly, students must pay attention to the conditions of Theorem 9.18.

In the next two examples we show the correct and incorrect applications of Theorem 9.18.

Example 9.10. The function

$$f(x) = \frac{\cos x}{\ln x} \quad \text{is the product or } \cos x \text{ and } \frac{1}{\ln x}.$$

We have

$$\lim_{x\to 1^+} \cos x = \cos 1 > 0$$

and by Example 9.8

$$\lim_{x\to 1^+} \frac{1}{\ln x} = \infty.$$

Hence, by (a) of part (A) of Theorem 9.18 (page 232),

$$\lim_{x\to 1^+} \frac{\cos x}{\ln x} = \infty.$$

It is <u>incorrect</u> to state the following.

$$\lim_{x\to 1^+} \frac{\cos x}{\ln x} = \lim_{x\to 1^+} \frac{1}{0^+} = \infty.$$

Division by zero is undefined, it is a bad habit to allow oneself these kind of "interpretations." □

Example 9.11. The function

$$f(x) = \frac{\cos \pi x}{\ln x} \quad \text{is the product or } \cos \pi x \text{ and } \frac{1}{\ln x}.$$

By Example 9.8

$$\lim_{x \to 1^+} \frac{1}{\ln x} = \infty \quad \text{and} \quad \lim_{x \to 1^+} \cos \pi x = -1 < 0.$$

Hence, by (b) of part (A) of Theorem 9.18 (page 232),

$$\lim_{x \to 1^+} \frac{\cos \pi x}{\ln x} = -\infty.$$

It is incorrect to state the following.

$$\lim_{x \to 1^+} \frac{\cos \pi x}{\ln x} = \frac{\cos \pi}{0^-} = -1(-\infty) = \infty.$$

The product of the infinity symbol is undefined. □

Theorem 9.18 (page 232) must be applied correctly. If the conditions of the theorem hold, then the conclusions. No more, no less.

Objectives

Students should be able to

1. evaluate infinite limits of quotient functions at a number.
2. find vertical asymptotes of quotient functions.

Each of the questions below tests the objectives of this section. Identify the objective(s) tested by each exercise. Make up your own exercises.

Exercises

22. Evaluate the limit
$$\lim_{x\to 1^-} \frac{\cos x}{\ln x}.$$

23. Find the vertical asymptotes of the function
$$\frac{x^3-1}{(x^2-1)(x+3)}.$$

24. Explain why the application of Theorem 9.18 below is incorrect. Give its correct application.
$$\begin{aligned}\lim_{x\to\pi^-} \frac{3x}{\cos(3x)} &= \lim_{x\to\pi^-} \frac{3\pi}{\cos(3\pi)}\\ &= \lim_{x\to\pi^-} \frac{3\pi}{0^+} = 3(\infty) = \infty.\end{aligned}$$

25. Give an example of a non-constant function f and a function g such that
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = -\infty.$$

Complete solutions are provided on page **??**.

Vertical Asymptotes of Composition of Functions

We apply particular cases of parts (a), (b), (c), (f) of Theorem 5.4 (page 114), to study the vertical asymptotes of composition of functions.

Theorem 9.19. *a. If* $\lim_{x\to V^\pm} g(x) = b^+$ *and* $\lim_{y\to b^+} f(y) = \pm\infty$*, then*
$$\lim_{x\to V^\pm} f(g(x)) = \pm\infty.$$

b. If $\lim_{x \to V^{\pm}} g(x) = b^{-}$ *and* $\lim_{y \to b^{-}} f(y) = \pm\infty$*, then*

$$\lim_{x \to V^{\pm}} f(g(x)) = \pm\infty.$$

c. If $\lim_{x \to V^{\pm}} g(x) = b$ *and* $\lim_{y \to b} f(y) = \pm\infty$*, then*

$$\lim_{x \to V^{\pm}} f(g(x)) = \pm\infty.$$

d. If $\lim_{x \to V^{\pm}} g(x) = \pm\infty$ *and* $\lim_{y \to \pm\infty} f(y) = \pm\infty$*, then*

$$\lim_{x \to V^{\pm}} f(g(x)) = \pm\infty.$$

If the conditions of Theorem 9.19 hold, then the vertical line $x = V$ is a vertical asymptote of the function $f(g(x))$.

Attention should be paid to the correct application of this theorem.

Example 9.12. Observe that in the evaluation of the limit

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right)^2$$

we have the composition $f \circ g(x)$ of the functions

$$g(x) = \frac{1}{x} \quad \text{and} \quad f(x) = x^2.$$

For these functions we have the limits

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to \infty} x^2 = \infty.$$

By part (d) of Theorem 9.19 we conclude that

$$\lim_{x \to 0^+} \left(\frac{1}{x}\right)^2 = \infty.$$

This is the correct application of Theorem 9.19. It is incorrect to state the following.

$$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^2 = (\infty)^2 = \infty.$$

The expression $(\infty)^2$ is meaningless. □

Example 9.13. In the evaluation of the limit

$$\lim_{x\to 1^+}\tan\left(\frac{\pi x}{2}\right)$$

we have the composition $f \circ g(x)$ of the functions

$$g(x) = \frac{\pi x}{2} \quad \text{and} \quad f(x) = \tan x.$$

For these functions we have the limits

$$\lim_{x\to 1^+}\frac{\pi x}{2} = \frac{\pi}{2}^+ \quad \text{and} \quad \lim_{x\to \pi/2^+}\tan x = -\infty.$$

By part (a) of Theorem 9.19 we conclude that

$$\lim_{x\to 1^+}\tan\left(\frac{\pi x}{2}\right) = \infty.$$

It is incorrect to state the following.

$$\lim_{x\to 1^+}\tan\left(\frac{\pi x}{2}\right) = \tan\left(\frac{\pi}{2}\right) = \infty.$$

The tangent function is undefined at $\pi/2$. □

Take note that only unbounded functions may have infinite limits.

Objectives

Students should be able to

1. determine whether the composition of functions may have a vertical asymptote.
2. determine the vertical asymptotes of the composition of functions.

Each of the questions below tests the objectives of this section. Identify the objective(s) tested by each exercise. Make up your own exercises.

Exercises

26. Explain why the function
$$f(x) = x^2 \sin x$$
does not have vertical asymptotes on its domain $\mathbb{R}$.

27. Give an example of an unbounded function f and a bounded function g such that their composition $f(g(x))$ is bounded.

28. Find a vertical asymptote of the function
$$f(x) = \sec\left(\frac{\pi x}{x+2}\right).$$

29. Evaluate the limit
$$\lim_{x\to 1^+} \ln(x^2 - 1).$$

30. Explain why the evaluation of the limit shown below is incorrect. Give its correct evaluation.
$$\begin{aligned}\lim_{x\to 1^+} \ln(-\sin(\pi x)) &= \ln(-\sin(\pi)) \\ &= \ln(0^+) = -\infty.\end{aligned}$$

Complete solutions are provided on page 356.

Chapter 10

Limits at Infinity:Horizontal Asymptotes

Visually, in Figures 2.2 to 2.4 (pages 34 to 34), the horizontal line $y = K$ is a horizontal asymptote of the function f in the positive direction, and in Figures 2.6 to 2.8 (pages 39 to 40), the horizontal line $y = K$ is a horizontal asymptote of the function f in the negative direction.

Horizontal asymptotes are determined by limits at infinity. That is, we must consider the behaviour of functions for very large and very small numbers in their domain.

In Chapter 2 we established the meaning of an infinitely large (small) number. Basically, a number is infinitely large (small) if on the real line, it is very far from zero on the right (left).

We write $x \to \infty$ to indicate the set of numbers which are infinitely large, and $x \to -\infty$ to indicate the set of numbers which are infinitely very small.

Let us to emphasize that neither $x \to \infty$ nor $x \to -\infty$ should be interpreted as "x *approaches* infinity or negative infinity." The symbols ∞ and $-\infty$ do not have a place on the real line, therefore a number on the real line cannot approach something which is nowhere.

Figure 10.1 shows the graph of a function f whose values are close to the number M for infinity large numbers and to the number L for infinity small numbers. **??**

The horizontal lines $y = M$ or $y = L$ are horizontal asymptotes of the function f, if its limit at infinity is M or its limit at negative infinity is L.

Intuitively, the number M is the limit of the function $f(x)$ for infinitely large x if

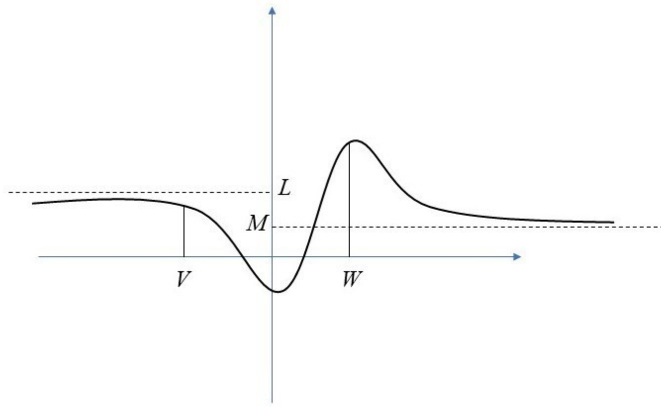


Figure 10.1: Graph of a function f with horizontal asymptotes

> we can make the values of $f(x)$ to be very close to M (as close as we like) by taking x infinitely large.

Similarly, the number L is the limit of the function $f(x)$ for infinitely small x if

> we can make the values of $f(x)$ to be very close to L (as close as we like) by taking x infinitely small.

These statements establish that the values $f(x)$ are close to a number for infinitely large or infinitely small numbers. Hence, $f(x)$ must be defined for infinitely large or small numbers. In other words, the domain of f contains infinite intervals of the form (a, ∞) or $(-\infty, a)$ for some number a. We then conclude that

> only functions whose domains contain infinite intervals may have horizontal asymptotes.

Examples of such functions are:

- sine and cosine functions.

- polynomial functions.
- rational functions.
- exponential functions.
- tangent inverse function.
- sine and cosine hyperbolic functions.

Students must be made aware that there are functions with horizontal asymptotes

- whose graph crosses the horizontal asymptote (see Figure 2.4 on page 34).
- which are discontinuous on an infinite interval (see Figure 2.5 on page 37).

Objectives

Students should be able to

1. determine *visually* whether a function has a horizontal asymptote in the positive or negative direction.

Each of the questions below tests the objectives of this section. Identify the objective(s) tested by each exercise. Make up your own exercises.

Exercises

1. Sketch the graph of a function which is discontinuous at infinitely many numbers and it has a horizontal asymptote in the negative direction.
2. Sketch the graph of an everywhere continuous function with horizontal asymptotes $y = 1$ and $y = -1$.
3. Sketch the graph of a nonconstant, everywhere continuous function with the horizontal asymptote $y = 2$ in both directions positive and negative.
4. Explain why the sine function does not have a horizontal asymptote.

Complete solutions are provided on page 358.

Horizontal Asymptotes of Quotient Functions

As in Chapter 7, the evaluation of finite limits at infinity of rational functions is based on the reciprocal function

$$F(x) = \frac{1}{x}$$ see Figure 9.18 on page 214.

This function has a horizontal asymptote $y = 0$ in the positive and negative directions. That is, the values of $F(x)$ get close to zero as the values of x increase and decrease infinitely. We express this behaviour with the limits

$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x} = 0.$$

The Law of Limits (Theorem 3.2 on page 47), holds for limits at infinity. Hence, by part (b) of this theorem, for any positive integer n

$$\lim_{x\to\pm\infty} \frac{1}{x^n} = \left[\lim_{x\to\pm\infty} \frac{1}{x}\right] \cdots \left[\lim_{x\to\pm\infty} \frac{1}{x}\right] = 0.$$

Moreover, for a function $g(x)$ with a finite limit at infinity

$$\lim_{x\to\pm\infty} g(x) = M$$

we have

$$\lim_{x\to\pm\infty} \frac{g(x)}{x^n} = \lim_{x\to\pm\infty} g(x) \left[\lim_{x\to\pm\infty} \frac{1}{x^n}\right] = M(0) = 0.$$

If

$$R(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + bx + b_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + ax + a_0}$$

is a rational function with $m \leq n$, then by factoring x^m from the numerator and x^n from the denominator, we have

$$R(x) = \frac{x^m \left(bm + \frac{b_{m-1}}{x} + \frac{b_{m-2}}{x^2} + \cdots + \frac{b}{x^{m-1}} + \frac{b_0}{x^m}\right)}{x^n \left(an + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n}\right)} \tag{10.1}$$

We apply the Laws of Limits to evaluate the limit at infinity of $R(x)$. You justify our conclusions.

$$\lim_{x\to\pm\infty} \frac{bm + \frac{b_{m-1}}{x} + \frac{b_{m-2}}{x^2} + \cdots + \frac{b}{x^{m-1}} + \frac{b_0}{x^m}}{an + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n}} = \frac{b_m}{a_n}.$$

Since either $n - m > 0$ or $m = n$

$$\lim_{x\to\pm\infty} \frac{x^m}{x^n} = \lim_{x\to\pm\infty} \frac{1}{x^{n-m}} = 0,$$

or

$$\lim_{x\to\pm\infty} \frac{x^m}{x^n} = \lim_{x\to\pm\infty} 1 = 1.$$

Therefore,

$$\lim_{x\to\pm\infty} R(x) = 0 \quad \text{if } m < n \tag{10.2}$$

and

$$\lim_{x\to\pm\infty} R(x) = \frac{b_m}{a_n} \quad \text{for } m = n. \tag{10.3}$$

As a particular case we have

$$\lim_{x\to\pm\infty} \frac{1}{a_n x^n + a_{n-1}x^{n-1} + \cdots + ax + a_0} = 0. \tag{10.4}$$

We define the limits at infinity and negative infinity from left and right as follows.

Definition 10.1. a. The limit at infinity of a function $f(x)$ is a number L from the right if

i. $\lim_{x\to\infty} f(x) = L$.

ii. If there is a positive number $V > 0$, such that

$$f(x) > L \quad \text{for any } x > V,$$

then $f(x) > L$ for $x \to \infty$ and we write

$$\lim_{x\to\infty} f(x) = L^+.$$

Figure A.9 (page 266) shows its graphic representation.

b. The limit at negative infinity of a function $f(x)$ is a number L from the right if

   i. $\lim_{x \to -\infty} f(x) = L$.

   ii. If there is a negative number $W < 0$, such that

$$f(x) > L \quad \text{for any } x < W,$$

   then $f(x) > L$ for $x \to -\infty$ and we write

$$\lim_{x \to -\infty} f(x) = L^{+}.$$

   Figure A.10 (page 267) shows its graphic representation.

c. The limit at infinity of a function $f(x)$ is a number L from the left if

   i. $\lim_{x \to \infty} f(x) = L$.

   ii. If there is a positive number $V > 0$, such that

$$f(x) < L \quad \text{for any } x > V,$$

   then $f(x) < L$ for $x \to \infty$ and we write

$$\lim_{x \to \infty} f(x) = L^{-}.$$

   Figure A.14 (page 269) shows its graphic representation.

d. The limit at negative infinity of a function $f(x)$ is a number L from the left if

   i. $\lim_{x \to -\infty} f(x) = L$.

   ii. If there is a negative number $W < 0$, such that

$$f(x) < L \quad \text{for any } x < W,$$

   then $f(x) < L$ for $x \to -\infty$ and we write

$$\lim_{x \to -\infty} f(x) = L^{-}.$$

   Figure A.15 (page 269) shows its graphic representation.

From Figure 9.18 (page 214), we see that

$$\lim_{x\to\infty} \frac{1}{x} = 0^+ \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x} = 0^-.$$

Example 10.1. By Equation (10.3),

$$\lim_{x\to-\infty} \frac{x^3+3x^2-12}{3x^3-3x^2+x} = \frac{1}{3}.$$

To determine if this limit is from the right or left we take a very small x, say $x = -10^6$, and evaluate the rational function

$$\frac{(-10^6)^3+3(-10^6)^2-12}{3(-10^6)^3-3(-10^6)^2+(-10^6)} = \frac{-10^{18}+3(10^{12})-12}{3(-10^{18})-3(10^{12})-10^6} > 0.$$

Hence,

$$\lim_{x\to-\infty} \frac{x^3+3x^2-12}{3x^3-3x^2+x} = \frac{1}{3}^+.$$

This is a first "approach" to determine that the limit is from the right. The derivative of this function should be applied to determine properly this same result. $\square$

Example 10.2. From the graph of the tangent inverse function we deduce that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}^-$$

and

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}^+.$$

$\square$

For the composition of functions we have parts (d) and (e) of Theorem 5.4 (page 114).

A. If $\quad \lim g(x) = \infty \qquad$ and $\qquad \lim\limits_{y \to \infty} f(y) = b$, then

$$\lim f(g(x)) = b.$$

B. If $\quad \lim g(x) = -\infty \qquad$ and $\qquad \lim\limits_{y \to -\infty} f(y) = b$, then

$$\lim f(g(x)) = b.$$

In the next two examples we show the correct and incorrect application of these two statements.

Example 10.3. To evaluate the limit

$$\lim_{x \to 0^+} \tan^{-1}(\ln x)$$

we consider the composition of the functions $\tan^{-1} y$ and $y = \ln x$. Thus,

$$\lim_{x \to 0^+} \ln x = -\infty \qquad \text{and} \qquad \lim_{y \to -\infty} \tan^{-1} y = -\frac{\pi}{2}.$$

By part (e) of Theorem 5.4

$$\lim_{x \to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}.$$

It is incorrect to evaluate this limit in the following manner

$$\lim_{x \to 0^+} \tan^{-1}(\ln x) = \tan^{-1}\left(\lim_{x \to 0^+} \ln x\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

because it does not make sense to evaluate a function at $-\infty$. $\square$

Example 10.4. To evaluate the limit

$$\lim_{x \to \pi^-} \frac{4}{\sec^2 x + \sec x - 4}$$

we consider the composition of the functions

$$\frac{4}{x^2 + x - 4} \qquad \text{and} \qquad y = \sec x.$$

Thus,

$$\lim_{x\to\pi^-} \sec x = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \frac{4}{x^2 + x - 4} = 0.$$

By part (d) of Theorem 5.4

$$\lim_{x\to\pi^-} \frac{4}{\sec^2 x + \sec x - 4} = 0.$$

It is incorrect to argue that because

$$\lim_{x\to\pi^-} \sec x = \infty$$

then

$$\lim_{x\to\pi^-} \frac{4}{\infty^2 + \infty - 4} - \frac{4}{\infty} = 0.$$

The symbol ∞ cannot be treated as a number. Moreover, the expression

$$\frac{4}{\infty} = 0 \quad \text{is incorrect.}$$

□

Objectives

Students should be able to

1. evaluate limits at infinity of rational functions.
2. evaluate limits at infinity of the quotient of functions.
3. evaluate limits at infinity of composition of functions.

Each of the questions below tests the objectives of this section. Review them and make up your own questions.

Exercises

5. Sketch the graph of one function whose limit at negative infinity is L from the right.

6. Explain why the limit

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

is neither from the right nor the left.

7. Evaluate the limits listed below and determine whether the limits are from the right or left.

   a. $\displaystyle\lim_{x\to\infty} \frac{3x^2-1}{x^6+2x-1}$

   b. $\displaystyle\lim_{x\to\infty} \frac{2x^2-x}{2x^2+2} - \frac{3x^2-7x^3}{x^3+2x^2+2}$

   c. $\displaystyle\lim_{x\to-\infty} \frac{x^4+3x^2-7}{x^6+x^3} + \frac{x^2-3x}{4-5x^2}$

8. Evaluate the limits listed below.

   a. $\displaystyle\lim_{x\to 0^-} \cot^{-1}(\csc x)$

   b. $\displaystyle\lim_{x\to 0^+} \ln(\sin x)$

Complete solutions are provided on page 360.

Infinite Limits at Infinity

As in the previous section we apply parts (e), (f), (g), (h), and (i) of Theorem 5.1 (page 101), to the product

$$f(x)\left(\frac{1}{g(x)}\right)$$

to evaluate limits such as

$$\lim_{x\to\pm\infty} f(x)g(x).$$

Theorem 10.2 is similar to Theorem 5.1.

Theorem 10.2. *A. If* $\displaystyle\lim_{x\to\pm\infty} f(x) = L \neq 0, \quad \lim_{x\to\pm\infty} \frac{1}{g(x)} = \infty,$

and $\displaystyle\lim_{x\uparrow K} \frac{1}{G(x)} = -\infty,$ *then*

$$a.\ \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \infty \quad \text{if } L > 0.$$

$$b.\ \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = -\infty \quad \text{if } L < 0.$$

$$c.\ \lim_{x\to\pm\infty} \frac{f(x)}{G(x)} = \infty \quad \text{if } L < 0.$$

$$d.\ \lim_{x\to\pm\infty} \frac{f(x)}{G(x)} = -\infty \quad \text{if } L > 0.$$

B. If $\displaystyle\lim_{x\to\pm\infty} f(x) = \infty, \quad \lim_{x\to\pm\infty} \frac{1}{g(x)} = L \neq 0,$
$\displaystyle\lim_{x\to\pm\infty} F(x) = -\infty,$ *then*

$$a.\ \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \infty \quad \text{if } L > 0.$$

$$b.\ \lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = -\infty \quad \text{if } L < 0.$$

$$c.\ \lim_{x\to\pm\infty} \frac{F(x)}{g(x)} = -\infty \quad \text{if } L > 0.$$

$$d.\ \lim_{x\to\pm\infty} \frac{F(x)}{g(x)} = \infty \quad \text{if } L < 0.$$

C. If $\lim_{x\to\pm\infty} f(x) = \infty$, *and* $\lim_{x\to\pm\infty} \frac{1}{g(x)} = \infty$, *then*

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \infty.$$

D. If $\lim_{x\to\pm\infty} f(x) = \infty$, *and* $\lim_{x\to\pm\infty} \frac{1}{g(x)} = -\infty$, *then*

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = -\infty.$$

E. If $\lim_{x\to\pm\infty} f(x) = -\infty$, *and* $\lim_{x\to\pm\infty} \frac{1}{g(x)} = \infty$, *then*

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = -\infty.$$

F. If $\lim_{x\to\pm\infty} f(x) = -\infty$, *and* $\lim_{x\to\pm\infty} \frac{1}{g(x)} = -\infty$, *then*

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \infty.$$

Example 10.5. We have the limits

$$\lim_{x\to\infty} x = \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{\sin(1/x)} = \lim_{t\to 0^+} \frac{1}{\sin(t)} = \infty.$$

By part (C) of Theorem 10.2

$$\lim_{x\to\infty} \frac{x}{\sin(1/x)} = \infty.$$

$\square$

Theorem 10.2 must be applied correctly avoiding products of the infinity symbol. If the conditions of the theorem hold, then the conclusion. No more, no less.

For rational functions

$$R(x) = \frac{b_m x^m + b_{m-1}x^{m-1} + \cdots + bx + b_0}{a_n x^n + a_{n-1}x^{n-1} + \cdots + ax + a_0}$$

with $m > n$ we have as in (10.1)

$$R(x) = \frac{x^m \left(bm + \frac{b_{m-1}}{x} + \frac{b_{m-2}}{x^2} + \cdots + \frac{b}{x^{m-1}} + \frac{b_0}{x^m}\right)}{x^n \left(an + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n}\right)}.$$

By the Laws of Limits

$$\lim_{x\to\pm\infty} \frac{bm + \frac{b_{m-1}}{x} + \frac{b_{m-2}}{x^2} + \cdots + \frac{b}{x^{m-1}} + \frac{b_0}{x^m}}{an + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n}} = \frac{b_m}{a_n}.$$

On the other hand,

$$\lim_{x\to\infty} \frac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty.$$

For $m - n$ odd

$$\lim_{x\to-\infty} \frac{x^m}{x^n} = \lim_{x\to-\infty} x^{m-n} = -\infty.$$

For $m - n$ even

$$\lim_{x\to-\infty} \frac{x^m}{x^n} = \lim_{x\to-\infty} x^{m-n} = \infty.$$

By parts (e) and (f) of Theorem 5.1 (page 101),

$$
\begin{aligned}
\lim_{x\to\infty} R(x) &= \lim_{x\to\infty}\left(\frac{x^m}{x^n}\right)\left(\frac{\left(bm+\frac{b_{m-1}}{x}+\frac{b_{m-2}}{x^2}+\cdots+\frac{b}{x^{m-1}}+\frac{b_0}{x^m}\right)}{\left(an+\frac{a_{n-1}}{x}+\frac{a_{n-2}}{x^2}+\cdots+\frac{a}{x^{n-1}}+\frac{a_0}{x^n}\right)}\right)\\
&= \begin{cases} \infty & : \frac{b_m}{a_n} > 0 \\ -\infty & : \frac{b_m}{a_n} < 0 \end{cases}
\end{aligned}
$$

For $m - n$ odd

$$
\begin{aligned}
\lim_{x\to-\infty} R(x) &= \lim_{x\to-\infty}\left(\frac{x^m}{x^n}\right)\left(\frac{\left(bm+\frac{b_{m-1}}{x}+\frac{b_{m-2}}{x^2}+\cdots+\frac{b}{x^{m-1}}+\frac{b_0}{x^m}\right)}{\left(an+\frac{a_{n-1}}{x}+\frac{a_{n-2}}{x^2}+\cdots+\frac{a}{x^{n-1}}+\frac{a_0}{x^n}\right)}\right)\\
&= \begin{cases} -\infty & : \frac{b_m}{a_n} > 0 \\ \infty & : \frac{b_m}{a_n} < 0 \end{cases}
\end{aligned}
$$

For $m - n$ even

$$
\begin{aligned}
\lim_{x\to-\infty} R(x) &= \lim_{x\to-\infty}\left(\frac{x^m}{x^n}\right)\left(\frac{\left(bm+\frac{b_{m-1}}{x}+\frac{b_{m-2}}{x^2}+\cdots+\frac{b}{x^{m-1}}+\frac{b_0}{x^m}\right)}{\left(an+\frac{a_{n-1}}{x}+\frac{a_{n-2}}{x^2}+\cdots+\frac{a}{x^{n-1}}+\frac{a_0}{x^n}\right)}\right)\\
&= \begin{cases} \infty & : \frac{b_m}{a_n} > 0 \\ -\infty & : \frac{b_m}{a_n} < 0 \end{cases}
\end{aligned}
$$

We conclude that rational functions with $m > n$ do not have horizontal asymptotes.

Example 10.6. For the limit of the rational function

$$
\lim_{x\to\infty} \frac{3x - 2x^2}{3x + 4}
$$

we have $m = 2, n = 1$ and $\dfrac{b_2}{a} = \dfrac{-2}{3} < 0$; hence, $m - n = 1$ is odd and

$$\lim_{x\to\infty} \frac{3x - 2x^2}{3x + 4} = -\infty.$$

$\square$

Example 10.7. For the limit of the rational function

$$\lim_{x\to-\infty} \frac{3x - 2x^4}{(3x + 4)(1 - 5x^2)}$$

we have $m = 4, n = 3$ and $\dfrac{b_4}{a_3} = \dfrac{-2}{-15} > 0$; hence,$m - n = 1$ is odd and

$$\lim_{x\to-\infty} \frac{3x - 2x^4}{(3x + 4)(1 - 5x^2)} = -\infty$$

$\square$

For a polynomial function $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + ax + a_0$, we factor x^n and obtain

$$p(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

By part (g) of Theorem 5.1 (page 101), we have

$$\lim_{x\to\infty} x^n = \infty \qquad \text{for any positive intern } n.$$

For n odd

$$\lim_{x\to-\infty} x^n = -\infty$$

and for n even

$$\lim_{x\to-\infty} x^n = \infty.$$

By (10.4)

$$\lim_{x\to\pm\infty} a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a}{x^{n-1}} + \frac{a_0}{x^n} = a_n.$$

By parts (e) and (f) of Theorem 5.1 (page 101),

$$\lim_{x\to\infty} p(x) = \infty \qquad \text{for any positive intern } n.$$

For n odd

$$\lim_{x\to-\infty} p(x) = \begin{cases} -\infty & : \quad a_n > 0 \\ \infty & : \quad a_n < 0 \end{cases}$$

For n even

$$\lim_{x\to-\infty} p(x) = \begin{cases} \infty & : \quad a_n > 0 \\ -\infty & : \quad a_n < 0 \end{cases}$$

For the composition of functions we have parts (a), (b), (c) and (f) of Theorem 5.4 (page 114).

Corollary 10.3. *a. If* $\lim g(x) = b^+$ *and* $\lim_{y\to b^+} f(y) = \pm\infty$, *then*

$$\lim f(g(x)) = \pm\infty.$$

b. If $\lim g(x) = b^-$ *and* $\lim_{y\to b^-} f(y) = \pm\infty$, *then*

$$\lim f(g(x)) = \pm\infty.$$

c. If $\lim g(x) = b$ *and* $\lim_{y\to b} f(y) = \pm\infty$, *then*

$$\lim f(g(x)) = \pm\infty.$$

d. If $\lim g(x) = \pm\infty$ *and* $\lim_{y\to\pm\infty} f(y) = \pm\infty$, *then*

$$\lim f(g(x)) = \pm\infty.$$

Example 10.8. To evaluate the limit

$$\lim_{x\to\infty} \csc\left(\frac{1}{x}\right)$$

we have the composition of the functions $\csc y$ and $y = \dfrac{1}{x}$, and the limits

$$\lim_{x\to\infty} \frac{1}{x} = 0^+ \quad \text{and} \quad \lim_{y\to 0^+} \csc y = \infty$$

Hence, by part (a) of Corollary 10.3

$$\lim_{x\to\infty} \csc\left(\frac{1}{x}\right) = \infty.$$

It is incorrect to evaluate this limit in the following manner

$$\lim_{x\to\infty} \csc\left(\frac{1}{x}\right) = \lim_{x\to\infty} \csc(0) = \infty.$$

We should never evaluate a function at a number which is not in its domain. □

Finally, we restate Theorem 5.1 (page 101). For the notation of Theorem 10.4 see Chapter 3.

Theorem 10.4. *If* $\lim a(x) = A$ *and* $\lim c(x) = C$ $(C \neq 0)$,

$\lim f(x) = \infty$, $\lim g(x) = \infty$,

$\lim u(x) = -\infty$, $\lim v(x) = -\infty$,

then

a. $\lim a(x) + f(x) = \infty$

b. $\lim a(x) + u(x) = -\infty$

c. $\lim f(x) + g(x) = \infty$

d. $\lim u(x) + v(x) = -\infty$

e. $\lim c(x)f(x) = \begin{cases} \infty & if \quad C > 0 \\ -\infty & if \quad C < 0. \end{cases}$

f. $\lim c(x)u(x) = \begin{cases} -\infty & if \quad C > 0 \\ \infty & if \quad C < 0. \end{cases}$

g. $\lim f(x)g(x) = \infty$

h. $\lim u(x)v(x) = \infty$

i. $\lim f(x)u(x) = -\infty$

> Theorem 10.4 must be applied correctly. If the conditions of the theorem hold, then the conclusions. No more, no less.

In the next example we show the correct and incorrect application of Theorem 10.4.

Example 10.9. Be Example 9.11 (page 235)

$$\lim_{x\to 1^+} \frac{\cos \pi x}{\ln x} = -\infty.$$

Hence, by part (h) of Theorem 10.4

$$\lim_{x\to 1^+} \left(\frac{\cos \pi x}{\ln x}\right)^2 = \lim_{x\to 1^+} \left(\frac{\cos \pi x}{\ln x}\right)\left(\frac{\cos \pi x}{\ln x}\right) = \infty.$$

It is incorrect to state the following.

$$\lim_{x\to 1^+} \left(\frac{\cos \pi x}{\ln x}\right)\left(\frac{\cos \pi x}{\ln x}\right) = (-\infty)(-\infty) = \infty.$$

The expression $(-\infty)(-\infty)$ is undefined. $\square$

Objectives

Students should be able to

1. evaluate limits at infinity of rational functions.
2. find horizontal asymptotes.

Exercises

9. Evaluate the limits listed below. If a limit does not exist explain why.

   a. $\lim_{x \to 1} \dfrac{\cos(\pi x)}{\sin(x-1)}$

   b. $\lim_{x \to 2^+} \dfrac{x^2}{x^3-8}$

   c. $\lim_{x \to 2^-} \dfrac{x^2}{x^3-8}$

10. Give two vertical asymptotes of the function

$$\frac{\cos(\pi x)}{\sin(x-1)}.$$

11. Explain why we do not apply Theorem 5.1 (page 257), to evaluate the limit

$$\lim_{x \to 0} \frac{x+1}{\sin\left(\frac{1}{x}\right)}.$$

12. Evaluate the limits listed below.

   a. $\lim_{x \to \infty} \ln\left(\dfrac{1}{x}\right)$

b. $\lim_{x \to -\infty} \frac{1-x}{e^x}$

13. Explain why the application of Theorem 10.2 below is incorrect and give the correct evaluation.

$$\begin{aligned} \lim_{x \to \pi/2^+} \frac{x - \pi}{\cos x} &= \lim_{x \to \pi/2^+} (x - \pi) \left[\frac{1}{\cos x} \right] \\ &= \lim_{x \to \pi/2^+} (x - \pi) \left[\lim_{x \to \pi/2^+} \frac{1}{\cos x} \right] \\ &= \left(-\frac{\pi}{2} \right) \frac{1}{0} = \left(-\frac{\pi}{2} \right) (\infty) = -\infty. \end{aligned}$$

Complete solutions are provided on page 363.

Summary

We presented the sequence of teaching of limits from basic functions.

1. **Functions.** Chapter 7.
   a. Basic functions on page 156.
   b. Transformations in Chapter 6 (page 131).
   c. Visual understanding of limits on page 169.
2. **Continuity** Chapter 8.
   a. Definition of continuous functions on pages 178, 180 and 183.
   b. Continuity of composition of functions on page 185.
3. **Limit Evaluation of Discontinuous Functions.** Chapter 8.
   a. Simplification techniques on page 189.

b. Finite limits from the right and left on page 195.

4. **Infinite limits** Chapter 9.

   a. Reciprocal functions on page 199.

   b. Bounded functions on page 220.

   c. Vertical asymptotes on pages 223, 232 and 236.

5. **Limits at infinity** Chapter 10.

   a. Horizontal asymptotes on page 245.

   b. Infinite limits on pages 251, 256 and 257.

6. **Squeeze Theorem** Theorem 3.13 (page 70).

7. **Trigonometric limits** Proposition 6.14 (page 138).

8. **L'Hospital's rule** Definition 6.15 (page 141).

9. **Limits of type $\infty - \infty$.** Theorem 6.19 (page 150).

10. **Limits of type $\infty/0$.** Theorem 6.19 (page 150).

11. **Limits of Type $0/\infty$.** Exercise 9 of Chapter 6.

Appendix A: Graphic Representations of Limits

In this appendix we provide the graphic representations of the limits studied in this *guide*. Read them following the order indicated:
$1 \to 2 \to 3 \to 4$.
See how the existence of $\delta > 0$ ($V > 0, U < 0$) follows *after* the arbitrary choice of $\varepsilon > 0$ ($M > 0, N < 0$).

Finite Limits at a Number

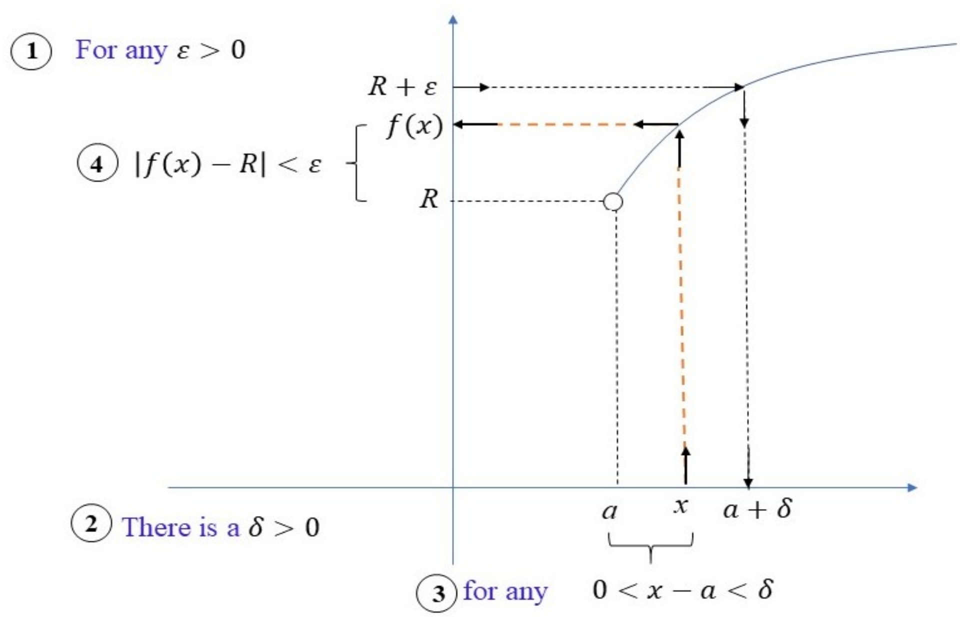


Figure A.1: $\lim_{x \to a^+} f(x) = R$

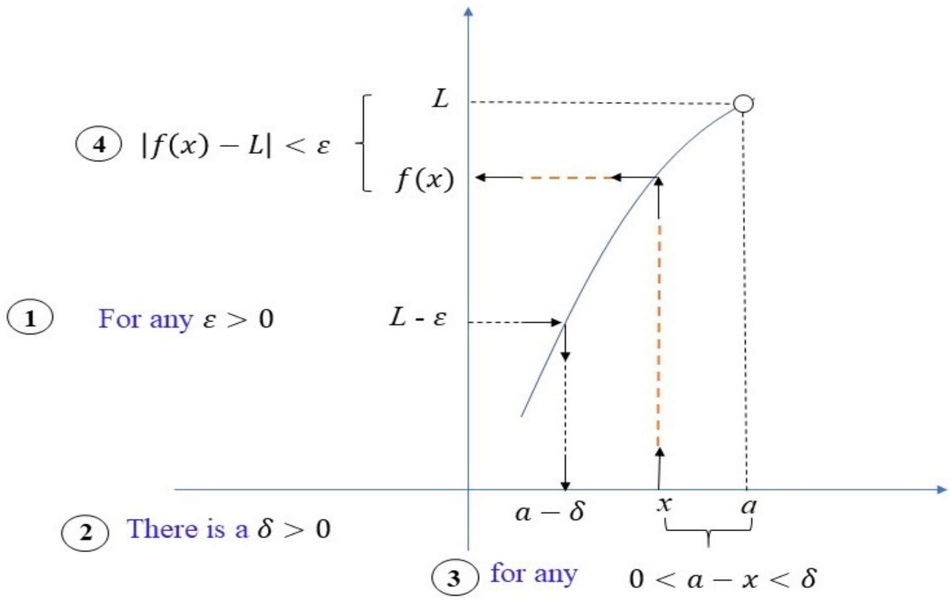


Figure A.2: $\lim_{x \to a^-} f(x) = L$

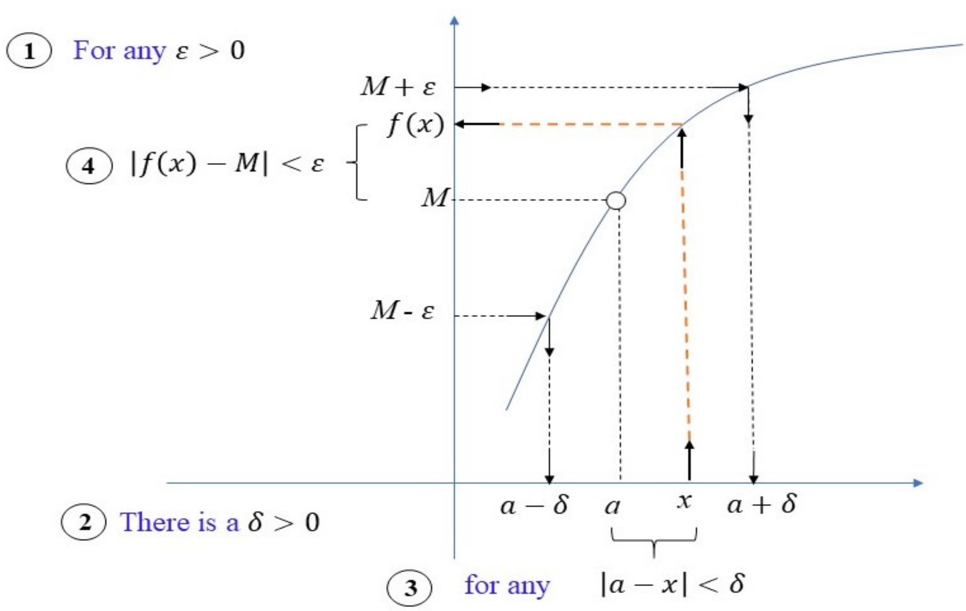


Figure A.3: $\lim_{x \to a} f(x) = M$

Finite Limits at Infinity

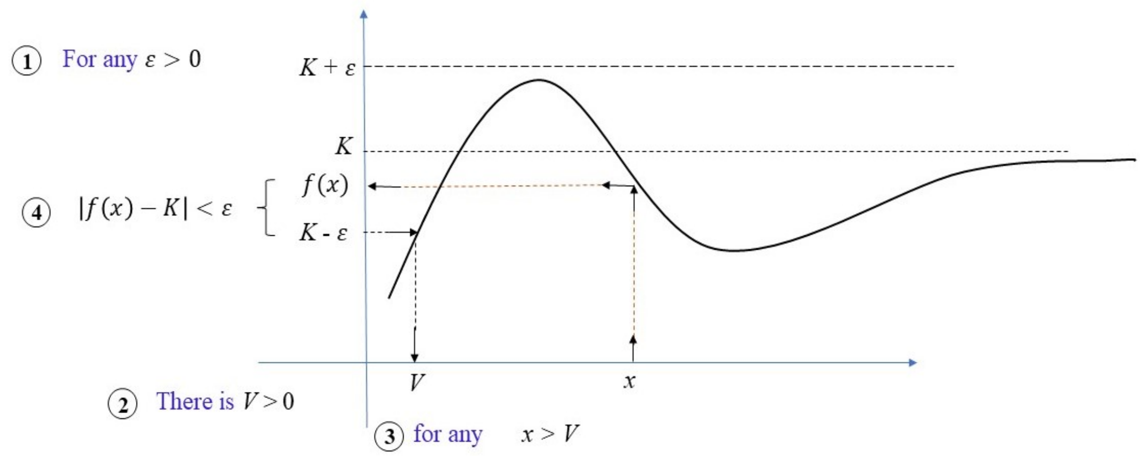


Figure A.4: $\lim_{x\to\infty} f(x) = K$

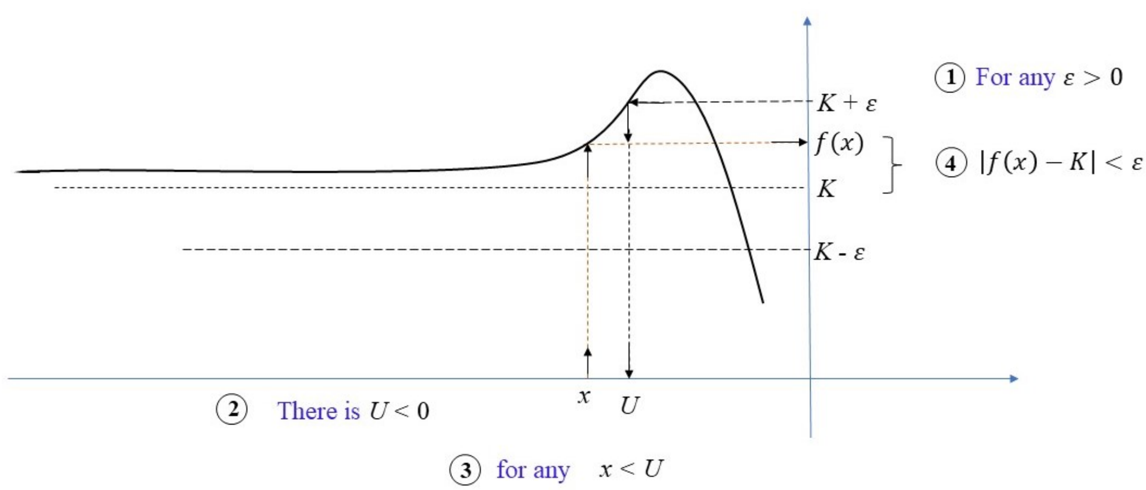


Figure A.5: $\lim_{x\to-\infty} f(x) = K$

Finite Limits from the Right

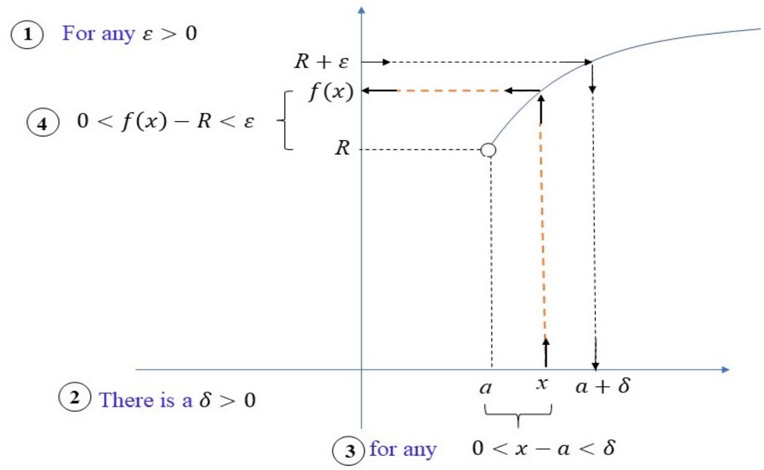


Figure A.6: $\lim_{x \to a^+} f(x) = R^+$

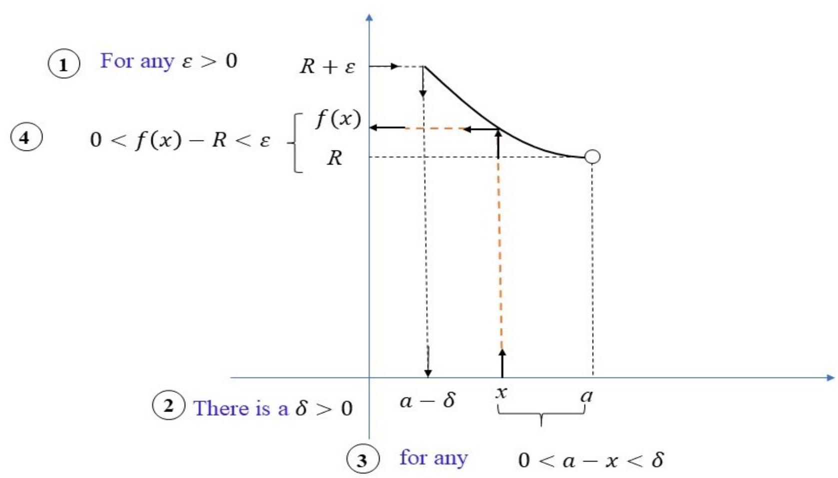


Figure A.7: $\lim_{x \to a^-} f(x) = R^+$

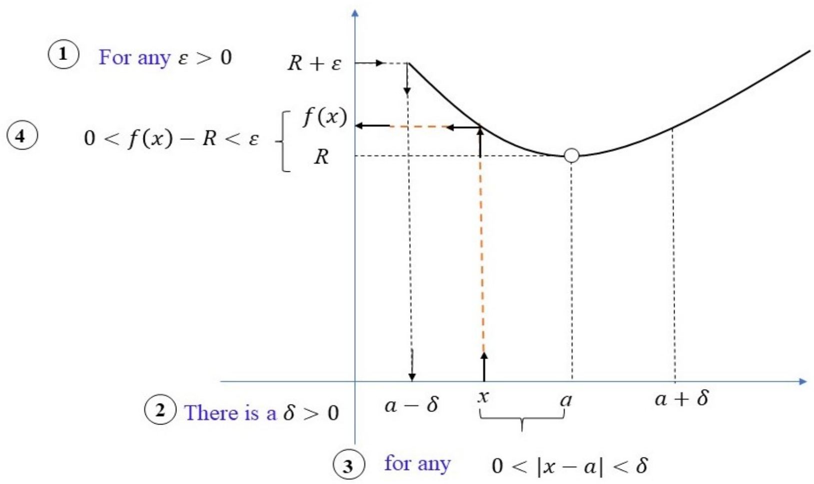


Figure A.8: $\lim_{x \to a} f(x) = R^+$

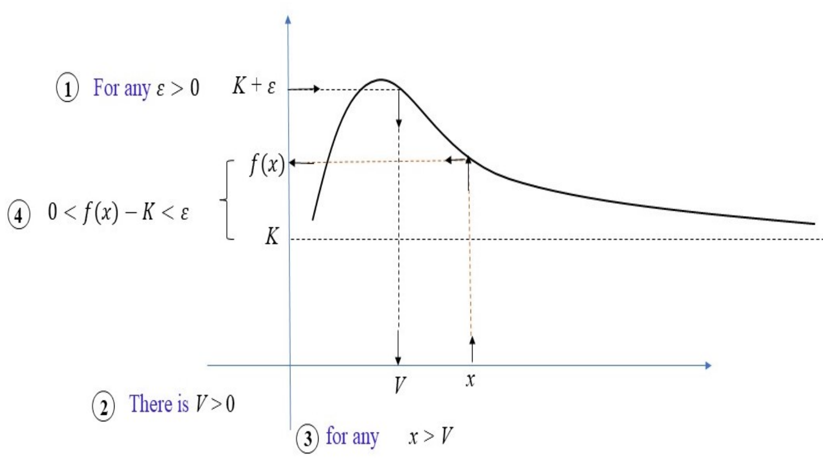


Figure A.9: $\lim_{x \to \infty} f(x) = R^+$

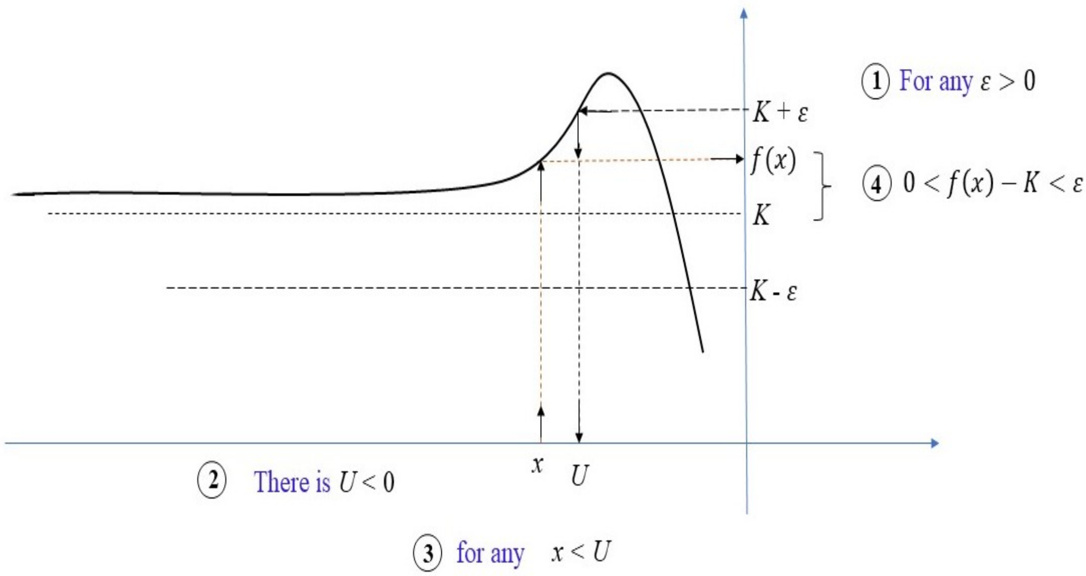


Figure A.10: $\lim_{x \to -\infty} f(x) = R^{+}$

Finite Limits from the Left

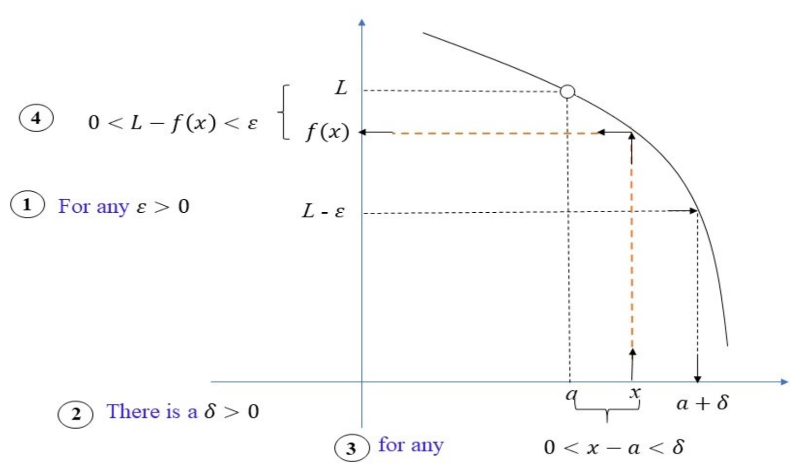


Figure A.11: $\lim_{x \to a^{+}} f(x) = L^{-}$

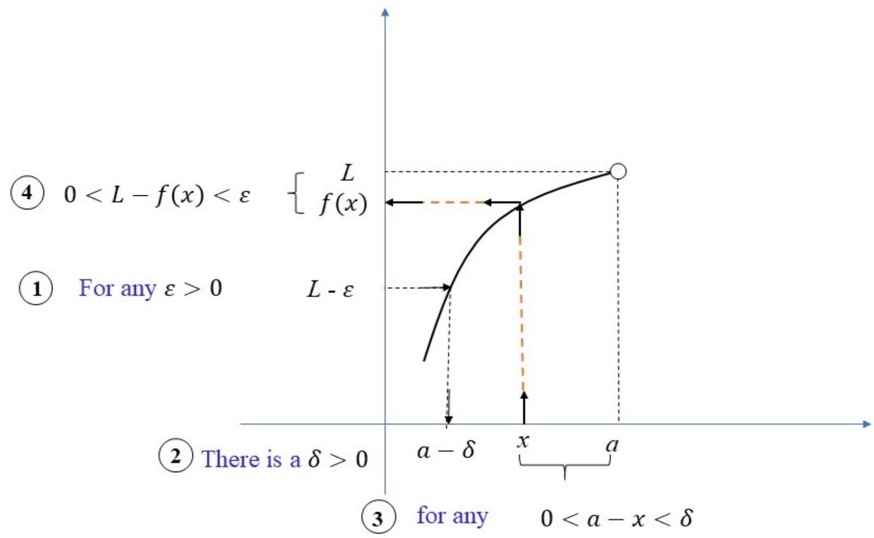


Figure A.12: $\lim_{x \to a^-} f(x) = L^-$

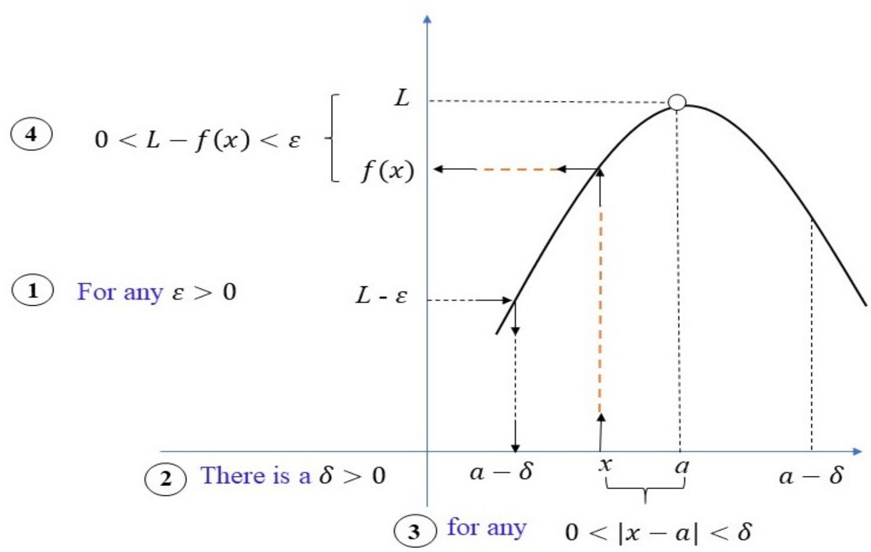


Figure A.13: $\lim_{x \to a} f(x) = L^-$

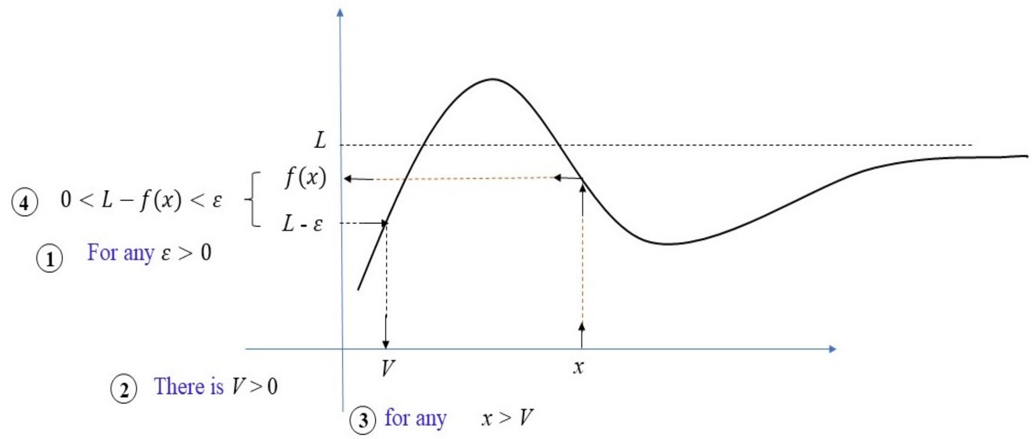


Figure A.14: $\lim_{x \to \infty} f(x) = L^-$

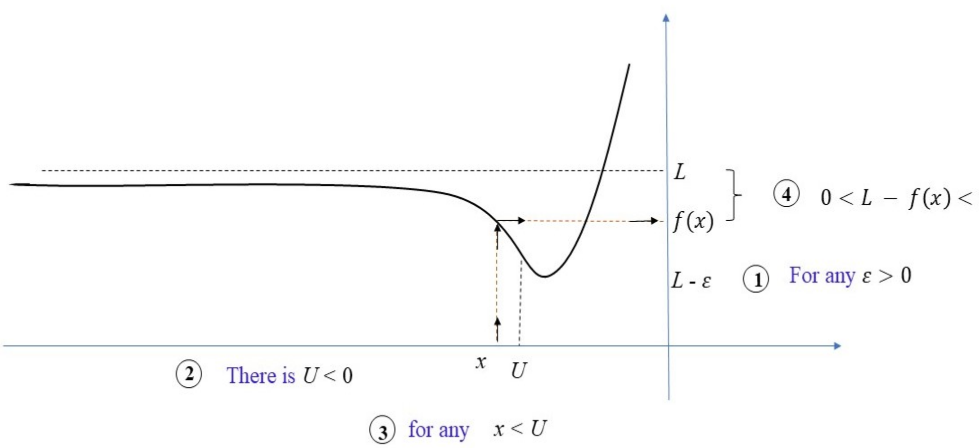


Figure A.15: $\lim_{x \to -\infty} f(x) = L^-$

Infinite Limits at a Number

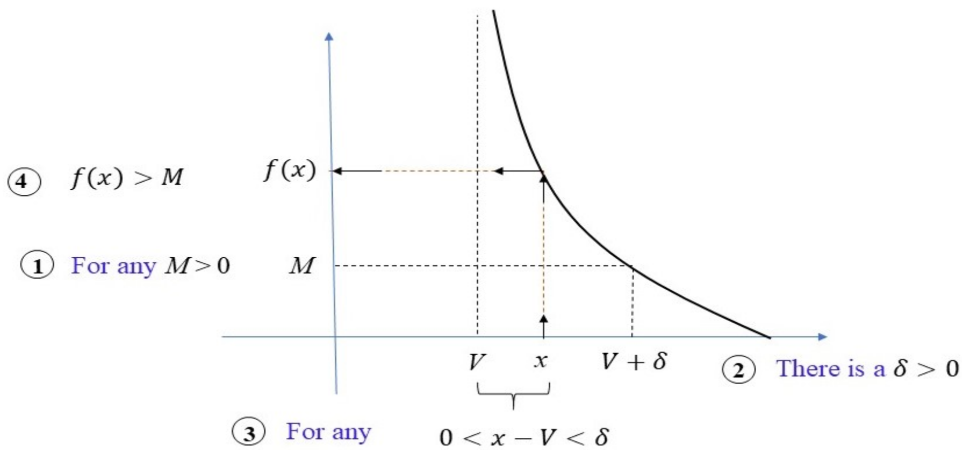


Figure A.16: $\lim\limits_{x \to V^+} f(x) = \infty$

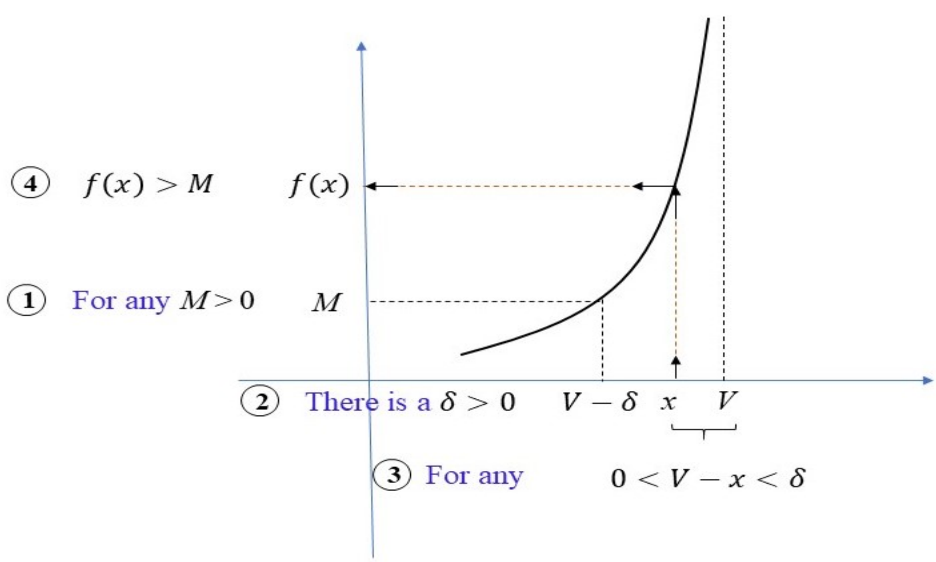


Figure A.17: $\lim\limits_{x \to V^-} f(x) = \infty$

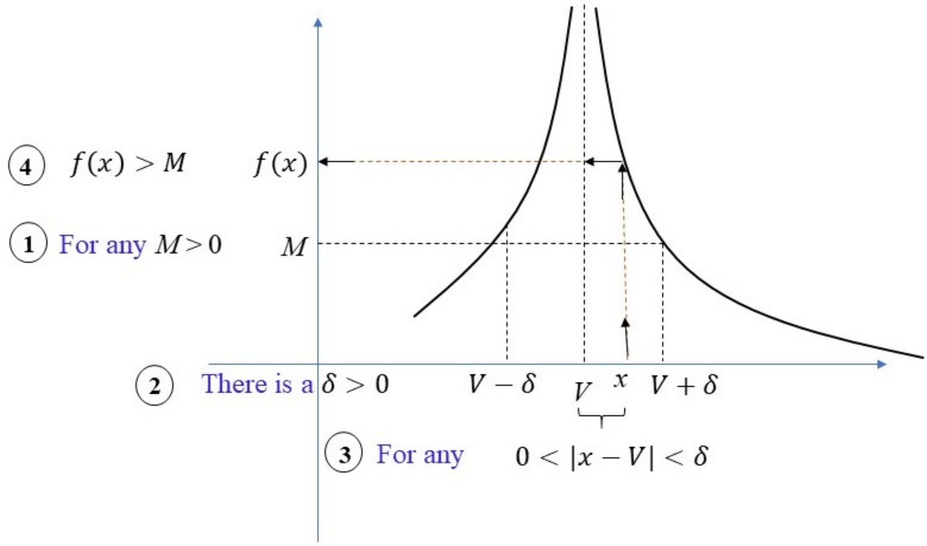


Figure A.18: $\lim\limits_{x \to V} f(x) = \infty$

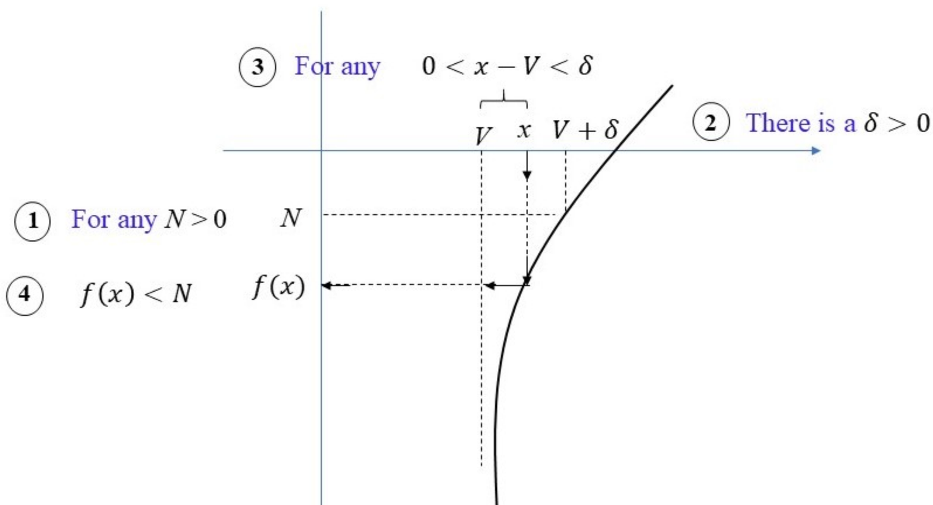


Figure A.19: $\lim\limits_{x \to V^+} f(x) = -\infty$

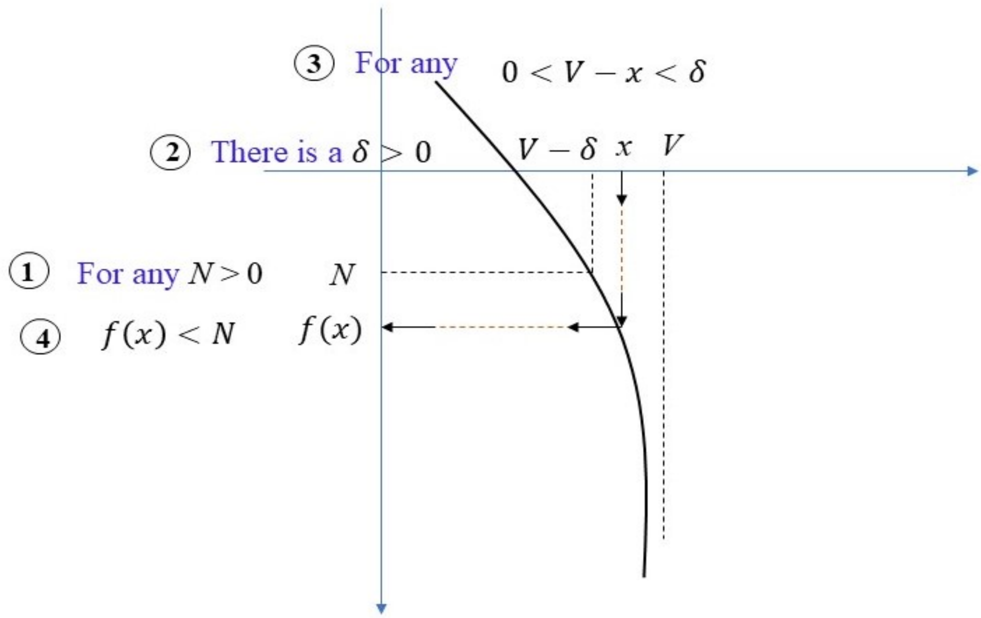


Figure A.20: $\lim\limits_{x \to V^-} f(x) = -\infty$

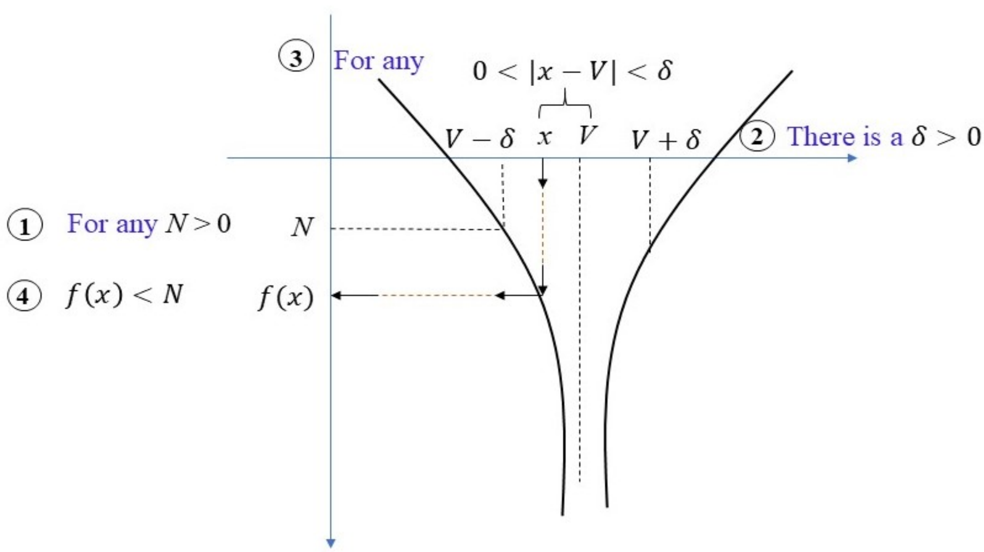


Figure A.21: $\lim\limits_{x \to V} f(x) = -\infty$

Infinite Limits at Infinity

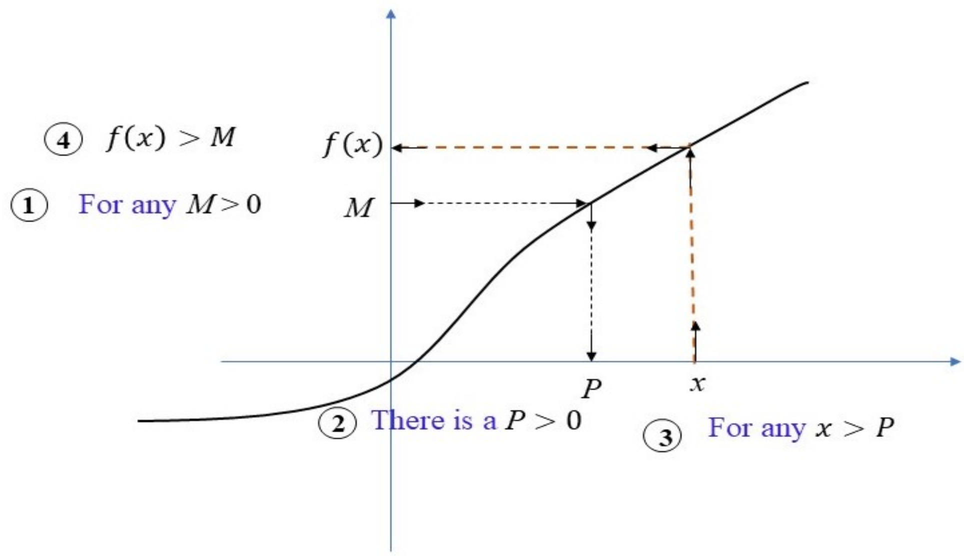


Figure A.22: $\lim_{x\to\infty} f(x) = \infty$

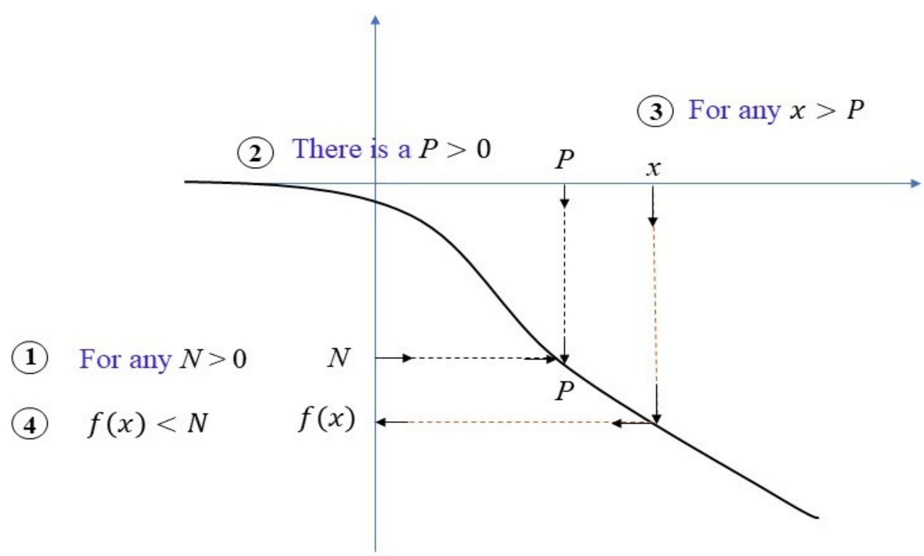


Figure A.23: $\lim_{x\to\infty} f(x) = -\infty$

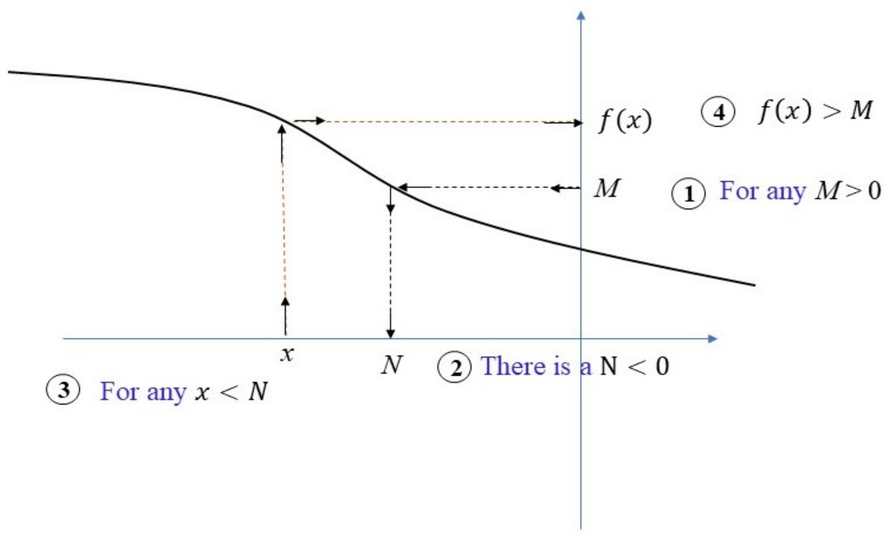


Figure A.24: $\lim_{x \to -\infty} f(x) = \infty$

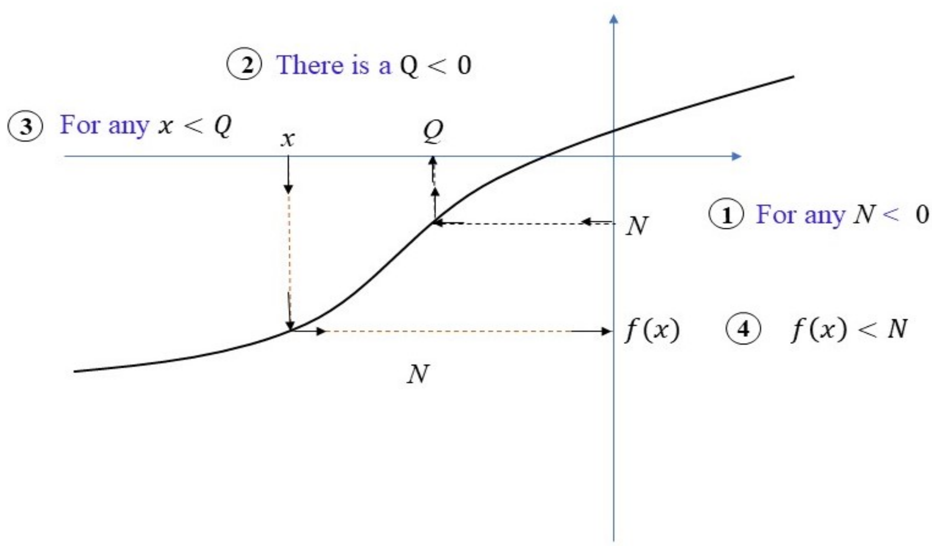


Figure A.25: $\lim_{x \to -\infty} f(x) = -\infty$

Appendix B: Solutions

In this appendix we provide the complete solutions to all the exercises in this *guide*.

Exercises Chapter I

Exercises I (page 26)

1. For a positive integer n, the number

$$a + \frac{1}{2n} \quad \text{belongs to the set } x \to a^+,$$

because

$$0 < a + \frac{1}{2n} - a = \frac{1}{2n} < \frac{1}{2}.$$

Hence, the set

$$\left\{ a + \frac{1}{2n} \mid n \in \mathbb{N} \right\} \quad \text{is infinite and it is a subset of} \quad x \to a^+.$$

2. a. For any positive number $\varepsilon > 0$, there is a number x such that $|x - 2| < \varepsilon$.

   b. $|a - b| < 10^{-6}$.

   c. For any positive number $\varepsilon > 0$ there is a rational number x such that $|x - \pi| < \varepsilon$.

3. For any $\delta > 0$, there is a negative integer k such that

$$k < -\frac{1}{2\delta\pi} < 0 \quad \Rightarrow \quad \delta < u = \frac{1}{2k\pi} < 0.$$

For this particular u we have

$$\sin\left(\frac{1}{u}\right) = \sin(2k\pi) = 0.$$

By part (b) of the negation of Definition 1.4 (page 8)

$$\sin\left(\frac{1}{x}\right) \quad \text{is neither positive nor negative for } x \to 0^-.$$

By part (c) of the negation of Definition 1.4.

$$\sin\left(\frac{1}{x}\right) \quad \text{is neither positive nor negative for } x \to 0^-.$$

4. For any $\delta > 0$, there is an integer n so that

$$n < -\frac{1}{2\delta\pi} < 0 \quad \Rightarrow \quad 0 < -\frac{1}{2n\pi} < \delta.$$

For this particular number $u = \dfrac{1}{2n\pi}$

$$\sin\left(\frac{1}{u}\right) = \sin(2n\pi) = 0 \quad \Rightarrow \quad \csc\left(\frac{1}{u}\right) \text{ is undefined.}$$

Hence, by part (b) of the negation of Definition 1.4 (page 8)

$$\csc\left(\frac{1}{x}\right) \quad \text{is undefined for } x \to 0^-.$$

5. The domain of the function

$$f(x) = \sqrt{1-x} \quad \text{is the interval } (-\infty, 1].$$

Hence,

$$\lim_{x \to 1^+} f(x) \quad \text{is undefined.}$$

6. By the negation of Definition 1.10 on page 21, the limit at zero from the left does not exist if for any real number L, there is a positive number $\varepsilon > 0$ such that for any positive number $\delta > 0$

$$\left|\sin\left(\frac{1}{x}\right) - L\right| \geq \varepsilon \quad \text{for some} \quad -\delta < x < 0. \tag{5}$$

For any positive number $\delta > 0$, there are negative numbers $N, M < 0$, such that

$$\left|\frac{2}{(4x-1)\pi}\right| < \delta \qquad \text{for any} \qquad x < N. \tag{6}$$

and

$$\left|\frac{6}{(12x-1)\pi}\right| < \delta \qquad \text{for any} \qquad x < M. \tag{7}$$

If the number $k < N$ is an integer, then

$$-\delta < u = \frac{2}{(4k-1)\pi} < 0 \quad \text{and} \quad \frac{1}{u} = \frac{(4k-1)\pi}{2} < 0.$$

For this particular number u we have

$$\sin\left(\frac{1}{u}\right) = \sin\left(\frac{(4k-1)\pi}{2}\right) = -1. \tag{8}$$

If L is any number we have by inequality 3 on page xii

$$\left|\sin\left(\frac{1}{x}\right) - L\right| \geq \left|\sin\left(\frac{1}{x}\right)\right| - |L| \tag{9}$$

$$\left|\sin\left(\frac{1}{x}\right) - L\right| \geq |L| - \left|\sin\left(\frac{1}{x}\right)\right| \tag{10}$$

For this number L we have three cases.
Case 1. If $|L| > 1$, we have the positive number $\quad \varepsilon = |L| - 1 > 0.$

By the statement (8) and inequality (10), we have that for this particular ε and any positive number $\delta > 0$ there is

$$-\delta < u = \frac{2}{(4k-1)\pi} < 0,$$

so that

$$\left|\sin\left(\frac{1}{u}\right) - L\right| \geq |L| - \left|\sin\left(\frac{1}{u}\right)\right| = |L| - 1 = \varepsilon.$$

Case 2. If $|L| < 1$, we have the positive number $\varepsilon = 1 - |L| > 0$.

By the statement (8) and inequality (9), we have that for this particular ε and any $\delta > 0$ there is

$$-\delta < u = \frac{2}{(4k+1)\pi} < 0,$$

so that

$$\left|\sin\left(\frac{1}{u}\right) - L\right| \geq \left|\sin\left(\frac{1}{u}\right)\right| - |L| = 1 - |L| = \varepsilon.$$

Similarly, if $k < M$ is an integer, then

$$-\delta < u = \frac{6}{(12k-1)\pi} < 0 \quad \text{and} \quad \frac{1}{u} = \frac{(12k-1)\pi}{6} < 0.$$

For this particular number u we have

$$\sin\left(\frac{1}{u}\right) = \sin\left(\frac{(12k-1)\pi}{6}\right) = -\frac{1}{2}. \tag{11}$$

Case 3. If $|L| = 1$, we have the positive number $\varepsilon = \dfrac{1}{2}$.

By the statement (11) and inequality (10), we have that for this ε and any positive number $\delta > 0$ there is

$$u = \frac{3}{(6k-1)\pi} < 0,$$

so that

$$\left|\sin\left(\frac{1}{u}\right) - L\right| \geq |L| - \left|\sin\left(\frac{1}{u}\right)\right| = 1 - \frac{1}{2} = \frac{1}{2} = \varepsilon.$$

In either case we have that the statement (5) holds. Hence, the limit

$$\lim_{x\to 0^-} \sin\left(\frac{1}{x}\right) \quad \text{does not exist.}$$

7. By Definition 1.11 (page 22) if

$$\lim_{x\to a} f(x) = L,$$

then for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for any} \quad 0 < |x - a| < \delta,$$

and if

$$\lim_{x\to a}(f(x) - L) = 0,$$

then for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - L| = |(f(x) - L) - 0| < \varepsilon \quad \text{for any} \quad 0 < |x - a| < \delta.$$

Since both statements are equal, it follows that

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a}(f(x) - L) = 0.$$

8. By the definition of absolute value (page xii)

$$||f(x)| - |L|| = \begin{cases} |f(x)| - |L| & \text{if} \quad |f(x)| - |L| > 0 \\ |L| - |f(x)| & \text{if} \quad |f(x)| - |L| < 0 \\ 0 & \text{if} \quad |f(x)| = |L| \end{cases}$$

By inequality 3 on page xii

$$|f(x)| - |L| \leq |f(x) - L| \quad \text{and} \quad |L| - |f(x)| \leq |f(x) - L|.$$

Therefore

$$||f(x)| - |L|| \leq |f(x) - L|. \tag{12}$$

Let $\varepsilon > 0$ be any positive number. By Definition 1.11 (page 22), there is a positive number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for any} \quad 0 < |x - a| < \delta.$$

Hence, by (12) for this same $\delta > 0$

$$||f(x)| - |L|| < \varepsilon \quad \text{for any} \quad 0 < |x - a| < \delta.$$

This proves that if $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} |f(x)| = |L|$.

9. Let $f(x)$ be the piecewise function

$$f(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

We have that $|f(x)| = 1$ for any nonzero x. Hence,

$$\lim_{x \to 0} |f(x)| = 1$$

and

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \lim_{x \to 0^-} f(x) = -1.$$

Therefore,

$$\lim_{x \to 0} f(x) \neq 1 \quad \text{moreover, this limit does not exist.}$$

10. Let $f(x)$ be the function

$$f(x) = \begin{cases} 0 & if \quad x < 0 \\ 1 & if \quad 0 \leq x < \pi. \end{cases}$$

It is clear that

$$\lim_{x \to 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x \to 0^+} f(x) = 1.$$

By the negation of Proposition 1.12 (page 1)

$$\lim_{x \to 0} f(x) \quad \text{does not exist.}$$

The function f is undefined for $x \to \pi^+$ because its domain is the interval $(-\infty, \pi)$. Hence, the function f is undefined for $x \to \pi$ and the limit

$$\lim_{x \to \pi} f(x) \quad \text{is undefined.}$$

11. a. Let $0 < \alpha, \theta < \dfrac{\pi}{2}$ be two numbers in the open interval $\left(0, \dfrac{\pi}{2}\right)$.

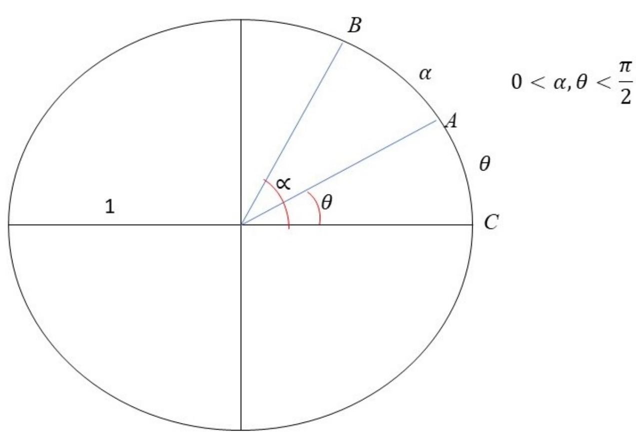


Figure B.1: Angles α and θ on the unit circle

On the unit circle, the arcs $AC = \theta$ and $BC = \alpha$ are shown in Figure B.1.

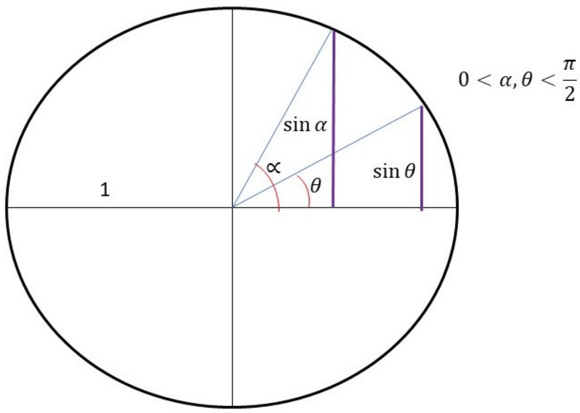


Figure B.2: $\sin\alpha$ and $\sin\theta$ on the unit circle

By the definition of the sine function, the purple vertical lines in Figure B.2 correspond to $\sin\theta$ and $\sin\alpha$ as shown.

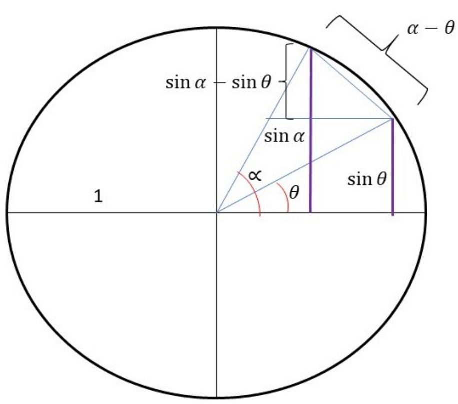


Figure B.3: Differences $\alpha - \theta$ and $\sin\alpha - \sin\theta$

The differences $\alpha - \theta$ and $\sin\alpha - \sin\theta$ are shown in Figure B.3.

The difference $\sin\alpha - \sin\theta$ is the side of a straight triangle, and the arc $\alpha - \theta$ is greater than its hypothenuse, say ab. Hence, we conclude that

$$|\sin\alpha - \sin\theta| \leq |ab| \leq |\alpha - \theta| \quad \text{for any } 0 < \alpha, \theta < \frac{\pi}{2}.$$

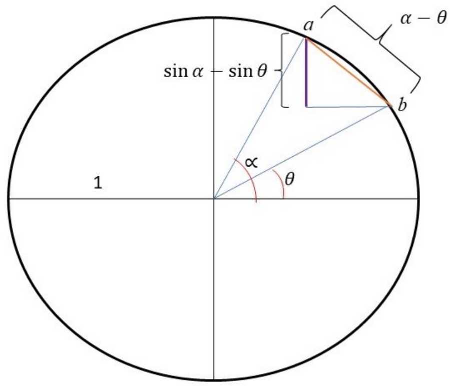


Figure B.4: Straight triangle with hypotenuse ab

b. For any $\varepsilon > 0$ be any positive number and let

$$0 < a < \frac{\pi}{2} \quad \text{be a number in the open interval } \left(0, \tfrac{\pi}{2}\right).$$

For the positive number

$$\delta = \min\left(\varepsilon, a, \frac{\pi}{2} - a\right) > 0,$$

we have that

$$\delta \leq a \quad \Rightarrow \quad a - \delta > 0 \tag{13}$$

and

$$\delta \leq \frac{\pi}{2} - a \quad \Rightarrow \quad \delta + a \leq \frac{\pi}{2}. \tag{14}$$

If $0 < |a - x| < \delta$, then $-\delta < x - a < \delta$; thus, by (13) and (14)

$$0 < a - \delta < x < \delta + a \leq \frac{\pi}{2}.$$

Hence, by part (a)

$$|\sin x - \sin a| \leq |a - x| < \delta < \varepsilon.$$

By Definition 1.11 (page 22), the sine function is continuous at a.

For any positive number $\varepsilon > 0$ there is the positive number $\delta = \min(\varepsilon, \pi/2) > 0$, so that

i. for any $0 < x < \delta < \dfrac{\pi}{2}$,

$$|\sin x - \sin 0| \leq |x| = x < \delta \leq \varepsilon \quad \text{by part (a).}$$

By Definition 1.9 (page 15), the sine function is continuous at 0 from the right.

ii. for any $0 < \dfrac{\pi}{2} - x < \delta \leq \dfrac{\pi}{2}$, then $0 < x < \dfrac{\pi}{2}$ and

$$|\sin x - \sin\left(\frac{\pi}{2}\right)| \leq \left|\frac{\pi}{2} - x\right| = \frac{\pi}{2} - x < \delta < \varepsilon \quad \text{by part (a).}$$

By Definition 1.10 (page 20), the sine function is continuous at $\dfrac{\pi}{2}$ from the left.

Therefore,

$$\lim_{x \to a} \sin x = \sin a \quad \text{for any } 0 \leq a \leq \frac{\pi}{2}.$$

Exercises Chapter II

Exercises II (page 43)

1. For any negative number $U < 0$ there is an integer $k < \dfrac{U}{2\pi}$. For this particular integer, the number

$$u = 2k\pi < U \quad \text{is so that} \quad \sin u = \sin(2k\pi) = 0.$$

In summary, for any $U < 0$ there is a numbers $u < U$ such that $\sin u = 0$.
Hence, by the negation of Definition 1.4 on page 8 the sine function is neither positive nor negative for $x \to -\infty$.

2. For any number K and any negative number $U < 0$, we have to show that there is a positive number $\varepsilon > 0$ so that (2.12) holds.

We consider two different cases for any number K.

[$K \neq 0$] For this number K there is the positive number $\varepsilon = |K| > 0$. Thus, for any negative number $U < 0$ there is an integer

$$k < \frac{2U + \pi}{4\pi} \quad \text{such that} \quad u = 2k\pi - \frac{\pi}{2} < U.$$

Hence,

$$|\cos u - K| = |K| = \varepsilon.$$

[$|K| = 0$] For this number K there is the positive number $\varepsilon = 1$. Thus, for any negative number $U < 0$, there is an integer

$$k < \frac{U}{2\pi} \quad \text{such that} \quad u = 2\pi k < U.$$

Hence,

$$|\cos u - K| = 1 = \varepsilon.$$

In any case (2.12) holds.

3. Let

$$\delta_C(x) = \begin{cases} x & if \quad x \in \mathbb{Z} \\ 0 & if \quad x \notin \mathbb{Z} \end{cases}$$

For any number K and any positive number $V > 0$, we have to show that there is a positive number $\varepsilon > 0$ such that (2.7) holds. We consider two different cases for any number K.

[$K \neq 0$] For this K there is the positive number $\varepsilon = |K| > 0$. Thus, for any $V > 0$, there is a non-integer $u > V$, and

$$|\delta_C(u) - K| = |K| = \varepsilon.$$

[$|K| = 0$] For this K there is the positive number $\varepsilon = 1$. Thus, for any $V > 0$ there is an integer $k > \max(V, 1)$, and

$$|\delta_C(k) - K| = |x| \geq 1 = \varepsilon.$$

Hence, the limit $\lim_{x\to\infty} \delta_C(x)$ does not exist.

4. Since $|\sin x| \le 1$, we have that for any non-zero number x

$$\left|\frac{\sin x}{x}\right| \le \frac{1}{|x|}.$$

For any positive number $\varepsilon > 0$, we have the positive number $V = \frac{1}{\varepsilon} > 0$, such that if $x > V$, then $\frac{1}{x} < \varepsilon$ and

$$\left|\frac{\sin x}{x}\right| \le \frac{1}{|x|} = \frac{1}{x} < \varepsilon.$$

Hence,

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0.$$

For any positive number $\varepsilon > 0$ we have the negative number $U = -\frac{1}{\varepsilon} < 0$, such that if $x < U$, then $0 < -\frac{1}{x} < \varepsilon$ and

$$\left|\frac{\sin x}{x}\right| \le \frac{1}{|x|} = -\frac{1}{x} < \varepsilon.$$

Hence,

$$\lim_{x\to-\infty} \frac{\sin x}{x} = 0.$$

Exercises Chapter III

Exercises III (page 71)

1. Let $f(x) = c$ be the constant function c and let a be any real number.

   a. For any positive number $\varepsilon > 0$ there is a positive number $\delta = 1 > 0$ such that

   $$|f(x) - c| = |c - c| = 0 < \varepsilon \quad \text{for every } 0 < |x - a| < 1.$$

Hence, $\lim_{x \to a} f(x) = c$ and by Proposition 1.12 (page 24)

$$\lim_{x \to a^+} f(x) = c \quad \text{and} \quad \lim_{x \to a^-} f(x) = c.$$

For any positive number $\varepsilon > 0$ there is a positive number $V = 1 > 0$ such that

$$|f(x) - c| = |c - c| = 0 < \varepsilon \quad \text{for every } x > V$$

Hence, $\lim_{x \to \infty} f(x) = c$.

b. For any positive number $\varepsilon > 0$ there is a negative number $U = -1 < 0$ such that

$$|f(x) - c| = |c - c| = 0 < \varepsilon \quad \text{for every } x < U$$

Hence, $\lim_{x \to -\infty} f(x) = c$.

2. Let $\varepsilon > 0$ be any positive number.

a. If

$$\lim_{x \to a^+} f(x) = M_1 \quad \text{and} \quad \lim_{x \to a^+} f(x) = M_2,$$

for the positive number $\varepsilon/2 > 0$, there are positive numbers $\delta_1, \delta_2 > 0$, such that

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad 0 < x - a < \delta_1$$

and

$$|f(x) - M_2| < \frac{\varepsilon}{2} \quad \text{for every} \quad 0 < x - a < \delta_2$$

These two statements hold for the positive number $\delta = \min(\delta_1, \delta_2)$ (see statement 1 on page 44). Thus, for any $0 < x - a < \delta$

$$\begin{aligned} |M_1 - M_2| &= |f(x) - M_2 + M_1 - f(x)| \\ &\leq |f(x) - M_1| + |f(x) - M_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε is any positive number, we conclude that $|M_1 - M_2| = 0$ and therefore $M_1 = M_2$.

b. If

$$\lim_{x \to a^-} f(x) = M_1 \quad \text{and} \quad \lim_{x \to a^-} f(x) = M_2,$$

then for the positive number $\varepsilon/2 > 0$, there are positive numbers $\delta_1, \delta_2 > 0$, such that

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad 0 < a - x < \delta_1$$

and

$$|f(x) - M_2| < \frac{\varepsilon}{2} \quad \text{for every} \quad 0 < a - x < \delta_2.$$

These two statements hold for the positive number $\delta = \min(\delta_1, \delta_2)$ (statement 1 on page 44)). Thus, for any $0 < a - x < \delta$

$$\begin{aligned} |M_1 - M_2| &= |f(x) - M_2 + M_1 - f(x)| \\ &\leq |f(x) - M_1| + |f(x) - M_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε is any positive number, we conclude that $M_1 = M_2$.

c. The proof for $x \to -\infty$ is similar. For the positive number $\varepsilon/2 > 0$, there are negative numbers $U_1 < 0$ and $U_2 < 0$ such that

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad x < U_1$$

and

$$|f(x) - M_1| < \frac{\varepsilon}{2} \quad \text{for every} \quad x < U_2.$$

These two statements hold for the negative number $U = \min(U_1, U_2)$ (statement 3 on page 44). Thus, for any $x < U$

$$\begin{aligned} |M_1 - M_2| &= |f(x) - M_2 + M_1 - f(x)| \leq |f(x) - M_1| + |f(x) - M_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Again, since ε is any positive number, we conclude that $M_1 = M_2$.

3. Let $\varepsilon > 0$ be any positive number.

   a. Part (a) of Theorem 3.2 (page 47) for $x \to a^+$.
   For the positive number $\varepsilon/2 > 0$ there are positive numbers $\delta_1, \delta_2 > 0$ such that

   $$|f(x) - K| < \frac{\varepsilon}{2} \qquad \text{for every} \qquad 0 < x - a < \delta_1, \tag{15}$$

   and

   $$|g(x) - M| < \frac{\varepsilon}{2} \qquad \text{for every} \qquad 0 < x - a < \delta_2. \tag{16}$$

   Both statements (15) and (16) hold for every $0 < x - a < \delta$ where $\delta = \min(\delta_1, \delta_2) > 0$. Thus, for every $0 < x - a < \delta$

   $$\begin{aligned}|(f(x) \pm g(x)) - (K \pm M)| &\leq |f(x) - K| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

   Hence, $\lim\limits_{x \to a^+} [f(x) \pm g(x)] = K \pm M$.

   b. Part (b) of Theorem 3.2 (page 47) for $x \to a^-$. We consider three cases.
   Case 1. If $K, M = 0$, for the positive number $\sqrt{\varepsilon} > 0$ there are positive numbers $\delta_1, \delta_2 > 0$ such that

   $$|f(x)| < \sqrt{\varepsilon} \qquad \text{for every} \qquad 0 < a - x < \delta_1, \tag{17}$$

   and

   $$|g(x)| < \sqrt{\varepsilon} \qquad \text{for every} \qquad 0 < a - x < \delta_2. \tag{18}$$

   Both statements (17) and (18) hold for every $0 < a - x < \delta$ where $\delta = \min(\delta_1, \delta_2) > 0$. Thus, for every $0 < a - x < \delta$

   $$\begin{aligned}|f(x)g(x)| &= |f(x)||g(x)| \\ &< \sqrt{\varepsilon}\sqrt{\varepsilon} = \varepsilon.\end{aligned}$$

Case 2. If $K = 0$ and $M \neq 0$, for the positive number

$$\frac{\varepsilon}{\varepsilon + |M|} > 0 \quad \text{there are positive numbers } \delta_1, \delta_2 > 0 \text{ such that}$$

$$|f(x)| < \frac{\varepsilon}{\varepsilon + |M|} \qquad \text{for every} \qquad 0 < a - x < \delta_1, \qquad (19)$$

and

$$|g(x)| - |M| \leq |g(x) - M| < \varepsilon \qquad \text{for every} \qquad 0 < a - x < \delta_2. \qquad (20)$$

Both statements (19) and (20) hold for every $0 < a - x < \delta$ where $\delta = \min(\delta_1, \delta_2) > 0$. Thus, for every $0 < a - x < \delta$

$$|f(x)g(x)| = |f(x)||g(x)| < \left[\frac{\varepsilon}{\varepsilon + |M|}\right](\varepsilon + M) = \frac{\varepsilon(\varepsilon + |M|)}{\varepsilon + |M|} = \varepsilon.$$

Case 3. If $K, M \neq 0$, for the positive number

$$\frac{\varepsilon}{2|M|} > 0, \quad \text{there is a positive number } \delta_1 > 0 \text{ such that}$$

$$|f(x) - K| < \frac{\varepsilon}{2|M|} \qquad \text{for every} \qquad 0 < a - x < \delta_1, \quad (21)$$

and

$$|f(x)| - |K| < \frac{\varepsilon}{2|M|}.$$

Then

$$|f(x)| < \frac{\varepsilon}{2|M|} + |K| = \frac{\varepsilon + 2|KM|}{2|M|} \qquad \text{for every} \qquad 0 < a - x < \delta_1. \qquad (22)$$

For the positive number

$$\frac{|M|\varepsilon}{\varepsilon + 2|KM|)} > 0, \quad \text{there is a positive number } \delta_2 \text{ such that}$$

$$|g(x) - M| < \frac{|M|\varepsilon}{\varepsilon + 2|KM|)} \qquad \text{for every} \qquad 0 < a - x < \delta_2. \tag{23}$$

By statement 1 on page 44, the statements (21), (22) and (23) hold for every $0 < a - x < \delta$ where $\delta = \min(\delta_1, \delta_2) > 0$ (). Thus, for every $0 < a - x < \delta$

$$\begin{aligned}
|f(x)g(x) - KM| &\leq |f(x) - f(x)M + f(x)M - KM| \\
&< |f(x)||g(x) - M| + |M||f(x) - K| \\
&< \left(\frac{\varepsilon + 2|KM|}{2|M|}\right)\left(\frac{|M|\varepsilon}{\varepsilon + 2|KM|)}\right) + \frac{|M|\varepsilon}{2|M|} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $\lim_{x \to a^-} [f(x)g(x)] = KM$.

c. Part (c) of Theorem 3.2 (page 47), for $x \to a^-$.
By Exercise 1, $\lim_{x \to a^-} c = c$ for any constant c. By part (b) of this exercise

$$cK = \left(\lim_{x \to a^-} c\right)\left(\lim_{x \to a^-} f(x)\right) = \lim_{x \to a^-} [cf(x)].$$

d. part (d) of Theorem 3.2 (page 47), for $x \to -\infty$.
For the positive number

$$\frac{\varepsilon|L|^2}{1 + |L|\varepsilon} > 0, \quad \text{there is a positive number } N < 0 \text{ such that}$$

$$|g(x) - L| < \frac{\varepsilon|L|^2}{1 + |L|\varepsilon} \qquad \text{for every} \qquad x < N. \tag{24}$$

Since

$$|M| - |g(x)| \leq |g(x) - L|\frac{\varepsilon|L|^2}{1 + |L|\varepsilon},$$

we have that

$$-|g(x)| < \frac{\varepsilon|L|^2}{1+|L|\varepsilon} - |L| = -\frac{|L|}{1+|L|\varepsilon}.$$

Therefore,

$$\frac{1}{|g(x)|} < \frac{1+|L|\varepsilon}{|L|} \qquad \text{for every} \qquad x < N. \tag{25}$$

Thus, for every $x < N$ the statement (25) holds and

$$\begin{aligned}\left|\frac{1}{g(x)} - \frac{1}{L}\right| &= \left|\frac{L-g(x)}{Lg(x)}\right| \\ &< \left(\frac{\varepsilon|L|^2}{1+|L|\varepsilon}\right)\left(\frac{1+|L|\varepsilon}{|L|^2}\right) = \varepsilon.\end{aligned}$$

Hence,

$$\lim_{x\to\infty}\frac{1}{g(x)} = \frac{1}{M} \quad \text{only if } M \neq 0.$$

e. Part (e) of Theorem 3.2 (page 47), for $x \to a^-$.
This follows from parts (c) and (d) assuming part (d) holds for $x \to a^-$.

$$\begin{aligned}\lim_{x\to a^-}\frac{f(x)}{g(x)} &= \left(\lim_{x\to a^-} f(x)\right)\left(\lim_{x\to a^-}\frac{1}{g(x)}\right) \\ &= K\left(\frac{1}{M}\right) = \frac{K}{M}.\end{aligned}$$

4. Let a be any number and $\varepsilon > 0$ be any positive number. For the identity function $f(x) = x$, we have the positive number $\delta = \varepsilon > 0$ such that

$$|f(x) - a| = |x - a| < \varepsilon \quad \text{for any } 0 < |x - a| < \delta.$$

Hence, $\lim\limits_{x\to a} x = a$.

5. Let a be any non zero number. By part (c) of Theorem 3.2 (page 47) and the previous exercise

$$\lim_{x \to a} \frac{1}{x} = \lim_{x \to a} \frac{1}{f(x)} = \frac{1}{a}.$$

6. Let $f(x) = \sin\left(\frac{1}{x}\right)$ and $g(x) = -\sin\left(\frac{1}{x}\right)$. Hence

$$f(x) + g(x) = \sin\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) = 0 \quad \text{for any nonero } x.$$

Thus,

$$\lim_{x \to 0}[f(x) + g(x)] = 0$$

and by Example 1.9 (page 17)

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \qquad \text{does not exist.}$$

7. We give the definitions of the expressions listed in Exercise 6.

   a. $f(x) \to R^{+}$ as $x \to a^{-}$.
   For any positive number $\varepsilon > 0$, there is a positive number $\delta > 0$ such that

   $$0 < f(x) - R < \varepsilon \quad \text{for every } 0 < a - x < \delta.$$

   b. $f(x) \to L^{-}$ as $x \to a^{+}$.
   For any positive number $\varepsilon > 0$, there is a positive number $\delta > 0$ such that

   $$0 < L - f(x) < \varepsilon \quad \text{for every } 0 < x - a < \delta.$$

   c. $f(x) \to L^{-}$ as $x \to \infty$.
   For any positive number $\varepsilon > 0$, there is a positive number $V > 0$ such that

   $$0 < L - f(x) < \varepsilon \quad \text{for every } x > V.$$

d. $f(x) \to L^-$ as $x \to -\infty$.
For any positive number $\varepsilon > 0$, there is a positive number $U < 0$ such that

$$0 < L - f(x) < \varepsilon \quad \text{for every } x < U.$$

8. The graphic representations of the definitions of Exercise 7 are in Appendix B.

a. Figure A.7 (page 265) shows the graphic representation of $\lim_{x \to a^-} f(x) = R^+$.

b. Figure A.11 (page 267) shows the graphic representation of $\lim_{x \to a^+} f(x) = L^-$.

c. Figure A.14 (page 269) shows the graphic representation of $\lim_{x \to \infty} f(x) = L^-$.

d. Figure A.15 (page 269) shows the graphic representation of $\lim_{x \to -\infty} f(x) = L^-$.

9. If either $c = 0$ or $K = 0$, then

$$\lim_{x \to a} [cf(x)] = \lim_{x \to a} 0 = 0 = (0)K.$$

Let $\varepsilon > 0$ be any positive number. If $c, K \neq 0$, then for the positive number

$$\frac{\varepsilon}{|c|} > 0, \quad \text{there is a positive number } \delta > 0 \text{ such that}$$

$$|f(x) - K| < \frac{\varepsilon}{|c|} \quad \text{for } 0 < |x - a| < \delta.$$

Hence,

$$|cf(x) - cK| = |c||f(x) - K| < |c| \left(\frac{\varepsilon}{|c|} \right) = \varepsilon \quad \text{for } 0 < |x - a| < \delta.$$

Therefore $\lim_{x \to a} [cf(x)] = cK$.

10. Negation of Definition 3.4 (page 56).

a. For any number R, there is a positive number $\varepsilon > 0$ for any positive number $\delta > 0$ such that either

$$g(x)-R \leq 0 \quad \text{or} \quad g(x)-R \geq \varepsilon \qquad \text{for some} \quad 0 < x-a < \delta.$$

Thus, either

- $\lim_{x\to a^+} g(x) \neq R$ for any real number R or
- $g(x) - R \not> 0$ for $x \to a^+$.

That is, either

$\lim_{x\to a^+} g(x)$ does not exist or $g(x) \not> R$ for $x \to a^+$.

b. For any number R, there is a positive number $\varepsilon > 0$ for any positive number $\delta > 0$ such that either

$$L-g(x) \leq 0 \quad \text{or} \quad L-g(x) \geq \varepsilon \qquad \text{for some} \quad 0 < a-x < \delta.$$

Thus, either

- $\lim_{x\to a^-} g(x) \neq L$ for any real number L or
- $L - g(x) \not> 0$ for $x \to a^-$.

That is, either

$\lim_{x\to a^-} g(x)$ does not exist or $g(x) \not< L$ for $x \to a^-$.

c. For any number R, there is a positive number $\varepsilon > 0$ for any positive number $\delta > 0$ such that either

$$g(x)-R \leq 0 \quad \text{or} \quad g(x)-R \geq \varepsilon \qquad \text{for some} \quad 0 < |x-a| < \delta.$$

Thus, either

- $\lim_{x\to a} g(x) \neq R$ for any real number R or
- $g(x) - R \not> 0$ for $x \to a$.

That is, either

$\lim_{x\to a} g(x)$ does not exist or $g(x) \not> R$ for $x \to a$.

d. For any number L, there is a positive number $\varepsilon > 0$ for any positive number $\delta > 0$ such that either

$$L - g(x) \le 0 \quad \text{or} \quad L - g(x) \ge \varepsilon \qquad \text{for some} \quad 0 < |x-a| < \delta.$$

Thus, either

- $\lim_{x\to a} g(x) \ne L$ for any real number L or
- $L - g(x) \not> 0$ for $x \to a$.

That is, either

$\lim_{x\to a} g(x)$ does not exist or $g(x) \not< L$ for $x \to a$.

e. For any number R, there is a positive number $\varepsilon > 0$ for any positive number $M > 0$ such that either

$$g(x) - R \le 0 \quad \text{or} \quad g(x) - R \ge \varepsilon \qquad \text{for some} \quad x > M.$$

Thus, either

- $\lim_{x\to\infty} g(x) \ne R$ for any real number R or
- $g(x) - R \not> 0$ for $x \to \infty$.

That is, either

$\lim_{x\to\infty} g(x)$ does not exist or $g(x) \not> R$ for $x \to \infty$.

f. For any number R, there is a positive number $\varepsilon > 0$ for any negative number $N < 0$ such that either

$$g(x) - R \le 0 \quad \text{or} \quad g(x) - R \ge \varepsilon \qquad \text{for some} \quad x < N.$$

Thus, either

- $\lim_{x\to-\infty} g(x) \ne R$ for any real number R or
- $g(x) - R \not> 0$ for $x \to -\infty$.

That is, either

$\lim_{x\to-\infty} g(x)$ does not exist or $g(x) \not> R$ for $x \to -\infty$.

11. Proof of part (a) of Theorem 3.7 (page 63).

a. Let $\lim_{x\to a^+} g(x) = b^+$ and $\lim_{y\to b^+} f(y) = c.$

For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|f(y) - c| < \varepsilon \quad \text{for every} \quad 0 < y - b < \delta_1.$$

For the positive number $\delta_1 > 0$, there is positive number $\delta_2 > 0$ such that

$$0 < g(x) - b < \delta_1 \quad \text{for every} \quad 0 < x - a < \delta_2.$$

Thus,

$$|f(g(x)) - c| < \varepsilon \quad \text{for every} \quad 0 < x - a < \delta_1.$$

Hence,

$$\lim_{x\to a^+} f(g(x)) = c.$$

b. Let $\lim_{x\to a^-} g(x) = b^+$ and $\lim_{y\to b^+} f(y) = c.$

For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|f(y) - c| < \varepsilon \quad \text{for every} \quad 0 < y - b < \delta_1.$$

For the positive number $\delta_1 > 0$, there is a positive number $\delta_2 > 0$ such that

$$0 < g(x) - b < \delta_1 \quad \text{for every} \quad 0 < a - x < \delta_2.$$

Thus,

$$|f(g(x)) - c| < \varepsilon \quad \text{for every} \quad 0 < a - x < \delta_1.$$

Hence,

$$\lim_{x\to a^-} f(g(x)) = c.$$

c. Let $\lim_{x\to -\infty} g(x) = b^{+}$ and $\lim_{y\to b^{+}} f(y) = c.$

For any positive number $\varepsilon > 0$ there is a positive number $\delta_1 > 0$ such that

$$|f(y) - c| < \varepsilon \quad \text{for every} \quad 0 < y - b < \delta_1.$$

For the positive number $\delta_1 > 0$, there is a negative number $U < 0$ such that

$$0 < g(x) - b < \delta_1 \quad \text{for every} \quad x < U.$$

Thus,

$$|f(g(x)) - c| < \varepsilon \quad \text{for every} \quad x < U.$$

Hence,

$$\lim_{x\to -\infty} f(g(x)) = c.$$

12. If $g(x) \to R^{+}$ as $x \to a$, then for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$0 \leq f(x) - R < \varepsilon \qquad \text{for every} \quad 0 < |x - a| < \delta.$$

Hence, for any positive number $\varepsilon > 0$ there is a positive number $\delta > 0$ such that

$$|f(x) - R| < \varepsilon \qquad \text{for every} \quad 0 < |x - a| < \delta.$$

Thus, $g(x) \to R$ as $x \to a$.

13. Let $h(x) = x \sin\left(\frac{1}{x}\right)$. By Example 3.12

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

and by Example 3.4 (page 60), for any positive number $\delta > 0$ there are integers $0 < a, b < \delta$ such that

$$h(a) > 0 \quad \text{and} \quad h(b) < 0.$$

Hence, $h(x) \not> 0$ for $x \to 0$ and by part (c) of Question 10,

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \neq 0^+.$$

14. Proof of Theorem 3.10 (page 68).

a. For $x \to a^+$. There is a positive number $\mu > 0$, such that

$$f(x) \leq 0 \quad \text{for every} \quad 0 < x - a < \mu.$$

For any positive number $\varepsilon > 0$, there is a positive number $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for every} \quad 0 < x - a < \delta_1.$$

If $\delta = \min(\mu, \delta_1) > 0$, then the above two inequalities hold for any every $0 < x - a < \delta$. Thus, $-\varepsilon < L - f(x) < \varepsilon$ and $f(x) < 0$. Therefore,

$$L = L - f(x) + f(x) \leq L - f(x) < \varepsilon.$$

Since ε is arbitrary, L cannot be positive. Hence, $L \leq 0$.

b. For $x \to a^-$. There is a positive number $\mu > 0$, such that

$$f(x) \leq 0 \quad \text{for every} \quad 0 < a - x < \mu.$$

For any positive number $\varepsilon > 0$, there is a positive number $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for every} \quad 0 < a - x < \delta_1.$$

If $\delta = \min(\mu, \delta_1) > 0$, then the above two inequalities hold for every $0 < a - x < \delta$. Thus, $-\varepsilon < L - f(x) < \varepsilon$ and $f(x) < 0$. Therefore,

$$L = L - f(x) + f(x) \leq L - f(x) < \varepsilon.$$

Since ε is arbitrary, L cannot be positive. Hence, $L \leq 0$.

c. For $x \to -\infty$. There is a negative number $N_1 < 0$, such that

$$f(x) \leq 0 \quad \text{for every} \quad x < N_1.$$

For any positive number $\varepsilon > 0$, there is a negative number $N_2 < 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for every} \quad x < N_2.$$

If $N = \min(N_1, N_2) > 0$, then the above two inequalities hold for every $x < N$. Thus, $-\varepsilon < L - f(x) < \varepsilon$ and $f(x) < 0$. Therefore,

$$L = L - f(x) + f(x) \leq L - f(x) < \varepsilon.$$

Since ε is arbitrary, L cannot be positive. Hence, $L \leq 0$.

15. For any $\delta > 0$ there is a number c so that

$$0 < c < \delta,$$

in particular for any non-zero z there is a c so that

$$0 < c < \frac{1}{z}.$$

By Theorem 3.13 (page 70) with $h(x) = 0$, $f(x) = c$ and $g(z) = \dfrac{1}{z}$, we have

$$\lim_{x\to\infty} h(x) = 0 = \lim_{z\to\infty} g(x).$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} c = c = 0.$$

Exercises Chapter IV

Exercises IV (page 99)

1. If the function f with domain D_f is bounded, then there is a constant B such that

$$-B \leq f(x) \leq B \quad \text{for every } x \in D_f.$$

Therefore, the function f is bounded with bounds B and $-B$ above and below, respectively.

If f is bounded above with bound B_1 and below with bound B_2, then

$$B_2 \leq f(x) \leq B_1 \quad \text{for every } x \in D_f.$$

If $B = \max(|B_1|, |B_2|)$, then $|B_2| \leq B$. Thus, $-B \leq -|B_2|$ and $|B_1| \leq B$. By inequality 6 (page xii)

$$-B \leq -|B_2| \leq B_2 \leq f(x) \leq B_1 \leq |B_1| \leq B \quad \text{for every } x \in D_f.$$

Hence, f is bounded with bound B.

2. Negation of Definition 4.1 (page 74).
Let D_f be the domain of a function f. The function f is

a. *unbounded above* if for any number B_U

$$f(x) > B_U \qquad \text{for some} \quad x \in D_f.$$

b. *unbounded below* if for any number B_L

$$f(x) < B_L \qquad \text{for some} \quad x \in D_f.$$

c. *unbounded* if for any number B

$$|f(x)| > B \qquad \text{for some} \quad x \in D_f.$$

3. a. Negation of part (a) of Definition 4.4 (page 81).
The *limit of a function f at the number V from the right is not infinite* if there is a positive number $M > 0$ such that for any positive number $\delta > 0$

$$f(x) \leq M \quad \text{for some} \quad 0 < x - V < \delta. \tag{26}$$

We write,

$$\lim_{x \to V^+} f(x) \neq \infty \qquad \text{or} \qquad f(x) \not\to \infty \quad \text{as} \quad x \to V^+.$$

b. Negation of part (b) of Definition 4.4 (page 81).
The *limit of a function* f *at the number* V *from the left is not infinite* if there is a positive number $M > 0$ such that for any positive number $\delta > 0$

$$f(x) \leq M \quad \text{for some} \quad 0 < V - x < \delta.$$

We write,

$$\lim_{x \to V^-} f(x) \neq \infty \qquad \text{or} \qquad f(x) \not\to \infty \quad \text{as} \quad x \to V^-.$$

4. Negation of Definition 4.5 (page 82).

a. The *limit of a function* f *at the number* V *from the right is not negative infinite* if there is a negative number $N < 0$ such that for any positive number $\delta > 0$

$$f(x) \geq N \quad \text{for some} \quad 0 < x - V < \delta.$$

We write,

$$\lim_{x \to V^+} f(x) \neq -\infty \qquad \text{or} \qquad f(x) \not\to -\infty \quad \text{as} \quad x \to V^+.$$

b. The *limit of a function* f *at the number* V *from the left is not negative infinite* if there is a negative number $N < 0$ such that for any positive number $\delta > 0$

$$f(x) \geq N \quad \text{for some} \quad 0 < V - x < \delta.$$

$$\lim_{x \to V^-} f(x) \neq -\infty \qquad \text{or} \qquad f(x) \not\to -\infty \quad \text{as} \quad x \to V^-.$$

c. The *limit of a function* f *at the number* V *is negative not infinite* if there is a negative number $N < 0$ such that for any $\delta > 0$

$$f(x) \geq N \quad \text{for some} \quad 0 < |x - V| < \delta.$$

$$\lim_{x \to V} f(x) \neq -\infty \qquad \text{or} \qquad f(x) \not\to -\infty \quad \text{as} \quad x \to V.$$

5. We apply Definition 4.9 (page 89) to prove that

$$\lim_{x\to\infty} x = \infty \quad \text{and} \quad \lim_{x\to-\infty} x = -\infty.$$

Let $f(x) = x$ be the identity function.
For any positive number $M > 0$, there is a positive number $P = M > 0$, such that if $x > P$, then

$$f(x) = x > M.$$

Hence,

$$\lim_{x\to\infty} x = \infty \quad \text{by part (a) of Definition 4.9 (page 89).}$$

For any negative number $N < 0$, there is a negative number $Q = N < 0$, such that if $x < Q$, then

$$f(x) = x < N.$$

Hence,

$$\lim_{x\to-\infty} x = -\infty \quad \text{by part (d) of Definition 4.9 (page 89).}$$

6. Let $\omega(x)$ be the function

$$\omega(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} - \{0\} \\ -x & \text{if} \quad x \in \mathbb{I} \end{cases}$$

   a. We apply Definition 1.11 (page 22) to prove that

$$\lim_{x\to 0} \omega(x) = 0.$$

   Since $|\omega(x)| = |x|$ for any nonzero x, for any positive number $\varepsilon > 0$ there is $\delta = \varepsilon$, so that

$$|\omega(x)| = |x| < \varepsilon \quad \text{for every } 0 < |x| < \delta.$$

b. We assume that

$$\lim_{x\to a^+} \omega(x) = K \quad \text{for some number } K.$$

i. For $a > 0$, we consider three cases.

Case 1: $K > 0$**.** For $\varepsilon = K/2 > 0$, and any $\delta > 0$, there is an irrational u such that $0 < a < u < a + \delta$. If

$$|\omega(u) - K| < \varepsilon \quad \Rightarrow \quad K - \varepsilon < \omega(u) = -u < \varepsilon + K.$$

Hence,

$$0 < \frac{K}{2} < -u < \frac{3K}{2} \quad \text{a contradiction, since } u > 0.$$

Therefore, for $\varepsilon = \dfrac{K}{2} > 0$, and any $\delta > 0$,

$$|\omega(u) - K| \geq \varepsilon \quad \text{for} \quad 0 < u - a < \delta.$$

Case 2: $K < 0$**.** For $\varepsilon = -K/2 > 0$, and any $\delta > 0$, there is a rational v such that $0 < a < v < a + \delta$. If

$$|\omega(v) - K| < \varepsilon \quad \Rightarrow \quad K - \varepsilon < \omega(v) = v < \varepsilon + K.$$

Hence,

$$\frac{3K}{2} < v < \frac{K}{2} < 0 \quad \text{a contradiction, since } v > 0.$$

Therefore, for $\varepsilon = -\dfrac{K}{2} > 0$, and any $\delta > 0$,

$$|\omega(v) - K| \geq \varepsilon \quad \text{for} \quad 0 < v - a < \delta.$$

Case 3: $K = 0$**.** For $\varepsilon = a/2$ and any $\delta > 0$, there is a rational z such that $0 < a < z < a + \delta$. If

$$|\omega(z)| < \varepsilon \quad \Rightarrow \quad -\frac{a}{2} < \omega(z) = z < \frac{a}{2}.$$

Hence,

$$-a < z < a \quad \text{a contradiction, since } a < z.$$

Therefore, for $\varepsilon = a/2$ and any $\delta > 0$,

$$|\omega(z)| \geq \varepsilon \quad \text{for} \quad 0 < z - a < \delta.$$

In any case, by the negation of Definition 1.9 on page 16

$$\lim_{x\to a^+} \omega(x) \neq K \quad \text{for any } K.$$

ii. For $a < 0$ again we consider three cases.

Case 1. $K > 0$**.** For $\varepsilon = K/2 > 0$, and any $\delta > 0$, we consider the number $m = \min(0, a+\delta)$. There is a rational u such that $a < u < m \leq a + \delta$. If

$$|\omega(u) - K| < \varepsilon \quad \Rightarrow \quad K - \varepsilon < u < \varepsilon + K.$$

Hence,

$$0 < \frac{K}{2} < u < \frac{3K}{2} \quad \text{a contradiction, since } u < 0.$$

Therefore, for $\varepsilon = \dfrac{K}{2} > 0$, and any $\delta > 0$,

$$|\omega(u) - K| \geq \varepsilon \quad \text{for} \quad 0 < u - a < \delta.$$

Case 2. $K < 0$**.** For $\varepsilon = -K/2 > 0$, and any $\delta > 0$, we consider the number $m = \min(0, a+\delta)$. There is an irrational v such that $a < v < m \leq a + \delta$. If

$$|\omega(v) - K| < \varepsilon \quad \Rightarrow \quad K - \varepsilon < -v < \varepsilon + K.$$

Hence,

$$\frac{3K}{2} < -v < \frac{K}{2} < 0 \quad \text{a contradiction, since } v < 0.$$

Therefore, For $\varepsilon = -\dfrac{K}{2} > 0$, and any $\delta > 0$,

$$|\omega(v) - K| \geq \varepsilon \quad \text{for} \quad 0 < v - a < \delta.$$

Case 3. $K = 0$**.** For $\varepsilon = a/2 < 0$ and any $\delta > 0$, we consider the number $m = \min(\varepsilon, a+\delta)$. There is a rational z such that $a < z < m \leq a + \delta$. If

$$|\omega(z)| < -\varepsilon \quad \Rightarrow \quad \varepsilon < z < -\varepsilon.$$

This is a contradiction because $z < m \leq \varepsilon$. Therefore, for $\varepsilon = a/2$ and any $\delta > 0$,

$$|\omega(z)| \geq \varepsilon \quad \text{for} \quad 0 < z - a < \delta.$$

In any case, by the negation of Definition 1.9 on page 16

$$\lim_{x \to a^+} \omega(x) \neq K \quad \text{for any } K.$$

We leave the case of $x \to a^-$ to the reader. By negation of Proposition 1.12 (page 25)

$$\lim_{x \to a} \omega(x) \neq K \quad \text{for any } K.$$

c. We apply parts (a) and (b) of the negation of Definition 4.9 on page 90.

i. Let $M = 1 > 0$ and let $P > 0$ be any positive number. Hence,

$$\omega(x) = -x \ngeq M \quad \text{for an irrational number } x > P > 0.$$

Thus

$$\lim_{x \to \infty} \omega(x) \neq \infty.$$

ii. Let $N = -1 < 0$ and let $P > 0$ be any positive number. Hence,

$$\omega(x) = x \nleq N \quad \text{for a rational number } x > P > 0.$$

Thus

$$\lim_{x \to \infty} \omega(x) \neq -\infty.$$

d. We apply parts (c) and (d) of the negation of Definition 4.9 on page 90.
Let $M = 1 > 0$ and let $Q < 0$ be any negative number. Hence,

$$\omega(x) = x \ngeq M \quad \text{for a rational number } x < Q < 0.$$

Thus

$$\lim_{x \to -\infty} \omega(x) \neq \infty.$$

e. Let $N = -1 < 0$ and let $Q < 0$ be any negative number. Hence,

$$\omega(x) = -x \nleq N \quad \text{for an irrational number } x < Q < 0.$$

Thus

$$\lim_{x \to -\infty} \omega(x) \neq -\infty.$$

7. Let $V > 0$ be any positive number. For the negative number $-V < 0$ there is $\delta > 0$ such that

$$f(x) < -V < 0 \quad \text{for every} \quad 0 < |x - a| < \delta.$$

Hence,

$$|f(x)| = -f(x) > V \quad \text{for every} \quad 0 < |x - a| < \delta.$$

By part (c) of Definition 4.4 (page 81)

$$\lim_{x \to a} |f(x)| = \infty.$$

8. We prove the contrapositive for $x \uparrow V$.

i. We assume that

$$\lim_{x \to V} f(x) = K \quad \text{for some } K.$$

For the positive number $\varepsilon = |K| + 1 > 0$, there is a positive number $\delta_1 > 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for every} \quad 0 < |x - V| < \delta_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$ we have that for every $0 < |x - V| < \delta_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K.$$

For any positive number $\delta > 0$, we have either $\delta < \delta_1$ or $\delta_1 \leq \delta$.

If $\delta < \delta_1$ and x is a number such that $|x - V| < \delta < \delta_1$, then $-(|K| + 1) + K < f(x)$.
If $\delta_1 \leq \delta$ and x is a number such that $0 < |x - V| < \delta_1 \leq \delta$, then $f(x) > -|K| - 1 + K \geq -2|K| - 1$ (see inequality 6).
In either case $f(x) > N = -(2|K| + 1)$ for some $|x - V| < \delta$.
Therefore $\lim\limits_{x \to V} f(x) \neq -\infty$.

ii. We assume that

$$\lim_{x \to V^+} f(x) = K \quad \text{for some } K.$$

For the positive number $\varepsilon = |K| + 1 > 0$, there is a positive number $\delta_1 > 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for every} \quad 0 < x - V < \delta_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$ we have that for every $0 < x - V < \delta_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K.$$

For any positive number $\delta > 0$, we have either $\delta < \delta_1$ or $\delta_1 \leq \delta$.
If $\delta < \delta_1$ and x is a number such that $x - V < \delta < \delta_1$, then $-(|K| + 1) + K < f(x)$.
If $\delta_1 \leq \delta$ and x is a number such that $0 < x - V < \delta_1 \leq \delta$, then $f(x) > -|K| - 1 + K \geq -2|K| - 1$ (see inequality 6).
In either case $f(x) > N = -(2|K| + 1)$ for some $x - V < \delta$.
Therefore $\lim\limits_{x \to V^+} f(x) \neq -\infty$.

iii. We assume that

$$\lim_{x \to V^-} f(x) = K \quad \text{for some } K.$$

For the positive number $\varepsilon = |K| + 1 > 0$, there is a positive number $\delta_1 > 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for every} \quad 0 < V - x < \delta_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$ we have that for every $0 < V - x < \delta_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K.$$

For any $\delta > 0$, we have either $\delta < \delta_1$ or $\delta_1 \leq \delta$.
If $\delta < \delta_1$ and x is a number such that $V - x < \delta < \delta_1$, then $-(|K| + 1) + K < f(x)$.
If $\delta_1 \leq \delta$ and x is a number such that $0 < V - x < \delta_1 \leq \delta$, then $f(x) > -|K| - 1 + K \geq -2|K| - 1$ (see inequality 6).
In either case

$$f(x) > N = -(2|K| + 1) \quad \text{for some } V - x < \delta.$$

Therefore $\lim_{x \to V^-} f(x) \neq -\infty$.

9. We prove the contrapositive. We assume that

$$\lim_{x \to -\infty} f(x) = K \quad \text{for some number } K.$$

For the positive number $\varepsilon = |K| + 1 > 0$, there is a negative number $N_1 < 0$ such that

$$|f(x) - K| < \varepsilon \quad \text{for every} \quad x < N_1.$$

Since, $-\varepsilon < f(x) - K < \varepsilon$ we have that for every $x < N_1$

$$-(|K| + 1) + K < f(x) < |K| + 1 + K.$$

For any negative number $N < 0$, we have that $N < N_1$ or $N \geq N_1$.

If $N < N_1$ and $x < N < N_1$, then $f(x) < 2|K| + 1$.

If $N \geq N_1$ and $x < N_1 \leq N$, then $f(x) < 2|K| + 1$.
In either case

$$f(x) < M = 2|K| + 1 \quad \text{for some } x < N.$$

Therefore $\lim_{x \to -\infty} f(x) \neq \infty$.

10. Part (a) of Theorem 4.13 (page 94)

i. For $x \to V^+$.
(⇒) Let $M > 0$ be any positive number.
By part (a) of Definition 3.4 (page 56), for the positive number $\varepsilon = \dfrac{1}{M} > 0$, there is a positive number $\delta > 0$ such that

$$0 < f(x) < \varepsilon \qquad \text{for every} \quad 0 < x - V < \delta.$$

Hence,

$$\frac{1}{f(x)} > \frac{1}{\varepsilon} = M \qquad \text{for every} \quad 0 < x - V < \delta.$$

By part (a) of Definition 4.4 (page 81)

$$\lim_{x \to V^+} \frac{1}{f(x)} = \infty.$$

(⇐) Let $\varepsilon > 0$ be any positive number. By part (a) of Definition 4.4 (page 81), for the positive number $M = \dfrac{1}{\varepsilon} > 0$, there is a positive number $\delta > 0$ such that

$$\frac{1}{f(x)} > M = \frac{1}{\varepsilon} > 0 \quad \text{for every} \quad 0 < x - V < \delta.$$

We then have that

$$f(x) > 0 \quad \text{and} \quad f(x) < \varepsilon \quad \text{for every } 0 < x - V < \delta.$$

Hence, by part (a) of Definition 3.4 (page 56)

$$\lim_{x \to V^+} f(x) = 0^+.$$

ii. For $x \to V^-$.
(⇒) Let $M > 0$ be any positive number.
By part (a) of Exercise 7 of Chapter III, for the positive number $\varepsilon = \dfrac{1}{M} > 0$, there is a positive number $\delta > 0$ such that

$$0 < f(x) < \varepsilon \qquad \text{for every} \quad 0 < V - x < \delta.$$

Hence,

$$\frac{1}{f(x)} > \frac{1}{\varepsilon} = M \qquad \text{for every} \quad 0 < V - x < \delta.$$

By part (a) of Definition 4.4 (page 81),

$$\lim_{x \to V^-} \frac{1}{f(x)} = \infty.$$

($\Leftarrow$) Let $\varepsilon > 0$ be any positive number.
By part (a) of Definition 4.4 (page 81), for the positive number $M = \dfrac{1}{\varepsilon} > 0$, there is a positive number $\delta > 0$ such that

$$\frac{1}{f(x)} > M = \frac{1}{\varepsilon} > 0 \quad \text{for every} \quad 0 < V - x < \delta.$$

We then have that

$$f(x) > 0 \quad \text{and} \quad f(x) < \varepsilon \quad \text{for every } 0 < V - x < \delta.$$

Hence, by part (a) of Exercise 7 of Chapter III,

$$\lim_{x \to V^-} f(x) = 0^+.$$

11. Part (b) of Theorem 4.13 (page 94).

    i. For $x \to V$.
    ($\Rightarrow$) Let $N < 0$ be any negative number. By part (d) of Definition 3.4 (page 56), for the positive number $\varepsilon = -\dfrac{1}{N} > 0$, there is a positive number $\delta > 0$ such that

    $$0 < 0 - f(x) < \varepsilon \qquad \text{for every} \quad 0 < |x - V| < \delta.$$

    Hence,

    $$\frac{1}{f(x)} < N \qquad \text{for every} \quad 0 < |x - V| < \delta.$$

By part (c) of Definition 4.5 (page 82)

$$\lim_{x \to V} \frac{1}{f(x)} = -\infty.$$

($\Leftarrow$) Let $\varepsilon > 0$ be any positive number. By part (c) of Definition 4.5 (page 82), for this negative number $N = -\dfrac{1}{\varepsilon} < 0$, there is a positive number $\delta > 0$ such that

$$\frac{1}{f(x)} < N = -\frac{1}{\varepsilon} < 0 \quad \text{for every} \quad 0 < |x - V| < \delta.$$

We then have that

$$f(x) < 0 \quad \text{and} \quad f(x) < \varepsilon \quad \text{for every } 0 < |x - V| < \delta.$$

Hence, part (d) of Definition 3.4 (page 56),

$$\lim_{x \to V} f(x) = 0^{-}.$$

ii. For $x \to -\infty$.
($\Rightarrow$) By part (d) of Exercise 7 (page 293), if $N < 0$ is any negative number, then for the positive number $\varepsilon = -\dfrac{1}{N} > 0$, there is a negative number $U < 0$ such that

$$0 < 0 - f(x) < \varepsilon \qquad \text{for every} \quad x < U.$$

Hence,

$$\frac{1}{f(x)} < -\frac{1}{\varepsilon} = N \qquad \text{for every} \quad x < U.$$

By part (d) of Definition 4.9 (page 89),

$$\lim_{x \to -\infty} \frac{1}{f(x)} = -\infty.$$

($\Leftarrow$) Let $\varepsilon > 0$ be any positive number. By part (d) of Definition 4.9 (page 89), for this negative number

$N = -\dfrac{1}{\varepsilon} < 0$, there is a negative number $Q < 0$ such that

$$\frac{1}{f(x)} < N = -\frac{1}{\varepsilon} < 0 \quad \text{for every} \quad x < Q.$$

We then have that

$$f(x) < 0 \quad \text{and} \quad f(x) < \varepsilon \quad \text{for every } x < Q.$$

Hence, by part (d) of Exercise 7 of Chapter III,

$$\lim_{x \to -\infty} f(x) = 0^-.$$

12. Let $\omega(x)$ be the function of Exercise 6 above, and

$$\Omega(x) = \frac{1}{\omega(x)}.$$

a. For the negative number $N = -1 < 0$ and any positive number $\delta > 0$, we have that

$$\frac{1}{\omega(x)} = \frac{1}{x} \not< N \quad \text{for a rational } 0 < x = |x| < \delta.$$

Thus, by Exercise 4 above

$$\lim_{x \to 0} \frac{1}{\omega(x)} \neq -\infty.$$

b. Let us assume that

$$\lim_{x \to 0} \frac{1}{\omega(x)} = K \quad \text{for some } K \neq 0.$$

Hence, by Theorem 3.2 (page 47)

$$\lim_{x \to 0} \Omega(x) = \frac{1}{K}.$$

But, we proved in part (a) of Exercise 6 above that this limit is equal to zero. This contradiction implies that $K = 0$. But in this case by the same Theorem 3.2

$$0 = \left[\lim_{x\to 0} \omega(x)\right] \left[\lim_{x\to 0} \frac{1}{\omega(x)}\right] = \lim_{x\to 0} \omega(x) \left(\frac{1}{\omega(x)}\right) = \lim_{x\to 0} 1 = 1.$$

This contradiction implies that

$$\lim_{x\to 0} \frac{1}{\omega(x)} \neq K \quad \text{for any } K.$$

Exercises Chapter V

Exercises V (page 118)

1. Part (f) of Theorem 5.1 (page 101) for $x \to V$.
   By part (c) of Theorem 3.2 (page 47)

   $$\lim_{x\to V} -c(x) = -C.$$

   Let $M > 0$ be any positive number. For the particular negative number $-M < 0$, there is a positive number $\delta > 0$ such that

   $$u(x) < -M \quad \text{for every} \quad 0 < |x - V| < \delta.$$

   Hence,

   $$-u(x) > M \quad \text{for every} \quad 0 < |x - V| < \delta.$$

   Therefore,

   $$\lim_{x\to V} -u(x) = \infty.$$

Hence, by part (e) of Theorem 5.1 (page 101)

$$\lim_{x\to V} c(x)u(x) = \lim_{x\to V} -c(x)(-u(x))$$
$$= \begin{cases} \infty & if \quad -C > 0 \\ -\infty & if \quad -C < 0. \end{cases}$$
$$= \begin{cases} \infty & if \quad C < 0 \\ -\infty & if \quad C > 0. \end{cases}$$

2. Part (g) of Theorem 5.1 (page 101) for $x \to \infty$.
Let $M > 0$ be any positive number. For the particular positive number $M_1 = \sqrt{M}$, there are positive numbers $P_1, P_2 > 0$ such that

$$f(x) > M_1 \quad \text{for every} \quad x > P_1$$

and

$$g(x) > M_1 \quad \text{for every} \quad x > P_2.$$

Both statements hold for $P = \max(P_1, P_2) > 0$; thus,

$$f(x)g(x) > g(x)M_1 > M_1^2 = M \quad \text{for every} \quad x > P.$$

Therefore,

$$\lim_{x\to\infty} f(x)g(x) = \infty.$$

3. Part (h) of Theorem 5.1 (page 101) for $x \to -\infty$.
Similarly to Exercise 1 above, it can be proved that

$$\lim_{x\to-\infty} -u(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} -v(x) = \infty.$$

Hence, by part (g) for $x \to -\infty$

$$\infty = \lim_{x\to-\infty} (-u(x))(-v(x)) = \lim_{x\to-\infty} u(x)v(x).$$

4. a. Part (a) of Theorem 5.3 (page 112), for $x \to V^+$.
For any positive number $M > 0$, there is a positive number $\delta_1 > 0$ such that

$$f(x) > M \quad \text{for every} \quad 0 < x - V < \delta_1.$$

Since $f(x) \leq g(x)$ for $x \to V^+$, there is a positive number $\delta_2 > 0$, such that

$$g(x) \geq f(x) \quad \text{for every} \quad 0 < x - V < \delta_2.$$

If $\delta = \min(\delta_1, \delta_2) > 0$, then both statements hold for $\delta > 0$, and

$$g(x) \geq f(x) > M \quad \text{for every} \quad 0 < x - V < \delta.$$

Therefore

$$\lim_{x \to V^+} g(x) = \infty.$$

b. Part (b) of Theorem 5.3 (page 112), for $x \to V^-$.
For any negative number $N < 0$, there is a positive number $\delta_1 > 0$ such that

$$g(x) < N \quad \text{for every} \quad 0 < V - x < \delta_1.$$

Since $f(x) \leq g(x)$ for $x \to V^-$, there is a positive number $\delta_2 > 0$, such that

$$g(x) \geq f(x) \quad \text{for every} \quad 0 < V - x < \delta_2.$$

If $\delta = \min(\delta_1, \delta_2) > 0$, then both statements hold for $\delta > 0$, and

$$f(x) \leq g(x) < N \quad \text{for every} \quad 0 < V - x < \delta.$$

Therefore

$$\lim_{x \to V^-} f(x) = -\infty.$$

c. Part (c) of Theorem 5.3 (page 112), for $x \to \infty$. For any positive number $M > 0$, there is a positive number $P_1 > 0$ such that

$$f(x) > M \quad \text{for every} \quad x > P_1.$$

Since $f(x) \leq g(x)$ for $x \to \infty$, there is a positive number $P_2 > 0$ such that

$$g(x) \geq f(x) \quad \text{for every} \quad x > P_2.$$

If $P = \max(P_1, P_2) > 0$, then both statements hold for P, and

$$g(x) \geq f(x) > M \quad \text{for every} \quad x > P.$$

Hence,

$$\lim_{x \to \infty} g(x) = \infty.$$

d. Part (d) of Theorem 5.3 (page 112), for $x \to -\infty$.
For any negative number $N < 0$, there is a negative number $Q_1 < 0$ such that

$$f(x) < N \quad \text{for every} \quad x < Q_1.$$

Since $f(x) \leq g(x)$ for $x \to -\infty$, there is a negative number $Q_2 < 0$ such that

$$g(x) \geq f(x) \quad \text{for every} \quad x < Q_2.$$

If $Q = \min(Q_1, Q_2) < 0$, then both statements hold for Q, and

$$g(x) \leq f(x) < N \quad \text{for every} \quad x < Q.$$

Hence,

$$\lim_{x \to -\infty} g(x) = -\infty.$$

5. Part (f) of Theorem 5.4 (page 114),

i. There are two parts for $x \to V$.

1. If $\lim_{x\to V} g(x) = \infty$ and $\lim_{y\to\infty} f(y) = \infty$, then

$$\lim_{x\to V} f(g(x)) = \infty.$$

Proof. Let $M > 0$ be any positive number. For this M there is a positive number $P_1 > 0$ such that

$$f(y) > M \quad \text{for every} \quad y > P_1.$$

For the positive number $P_1 > 0$, there is a positive number $\delta > 0$ such that

$$g(x) > P_1 \quad \text{for every} \quad 0 < |x - V| < \delta.$$

Hence,

$$f(g(x)) > M \quad \text{for every} \quad 0 < |x - V| < \delta$$

and $\lim_{x\to V} f(g(x)) = \infty$. Q.E.D.

2. If $\lim_{x\to V} g(x) = -\infty$ and $\lim_{y\to-\infty} f(y) = -\infty$, then

$$\lim_{x\to V} f(g(x)) = -\infty.$$

Proof. Let $N < 0$ be any negative number. For this N there is a negative number $N_1 < 0$ such that

$$f(y) < N \quad \text{for every} \quad y < N_1.$$

For this negative number N_1 there is a positive number $\delta > 0$ such that

$$g(x) < N_1 \quad \text{for every} \quad 0 < |x - V| < \delta.$$

Hence,

$$f(g(x)) < N \quad \text{for every} \quad 0 < |x - V| < \delta,$$

and $\lim_{x\to V} f(g(x)) = -\infty$. Q.E.D.

ii. There are also two parts for $x \to -\infty$.

1. If $\lim_{x\to-\infty} g(x) = \infty$ and $\lim_{y\to\infty} f(y) = \infty$, then

$$\lim_{x\to-\infty} f(g(x)) = \infty.$$

Proof. Let $M > 0$ be any positive number. For this M there is a positive number $P_1 > 0$ such that

$$f(y) > M \quad \text{for every} \quad y > P_1.$$

For the positive number P_1, there is a negative number $Q < 0$ such that

$$g(x) > P_1 \quad \text{for every} \quad x < Q.$$

Hence,

$$f(g(x)) > M \quad \text{for every} \quad x < Q,$$

and $\lim_{x\to-\infty} f(g(x)) = \infty$. Q.E.D.

2. If $\lim_{x\to-\infty} g(x) = -\infty$ and $\lim_{y\to-\infty} f(y) = -\infty$, then

$$\lim_{x\to-\infty} f(g(x)) = -\infty.$$

Proof. Let $N < 0$ be any negative number. For this N there is a negative number $N_1 < 0$ such that

$$f(y) < N \quad \text{for every} \quad y < N_1.$$

For this negative number N_1 there is a negative number $Q < 0$ such that

$$g(x) < N_1 \quad \text{for every} \quad x < Q.$$

Hence,

$$f(g(x)) < N \quad \text{for every} \quad x < Q$$

and $\lim_{x\to-\infty} f(g(x)) = -\infty$. Q.E.D.

6. Both conditionals are true.

Proof. a. Let $M > 0$ be any positive number. For this M, there is a negative number $N < 0$ such that

$$f(y) > M \quad \text{for every} \quad y < N.$$

For this negative number $N < 0$, there is a positive number $P > 0$ such that

$$g(x) < N \quad \text{for every} \quad x > P.$$

Hence,

$$f(g(x)) > M \quad \text{for every} \quad x > P.$$

Therefore $\lim_{x\to\infty} f(g(x)) = \infty$.

b. Let $N < 0$ be any negative number. For this negative number N, there is a positive number $P > 0$ such that

$$f(y) < N \quad \text{for every} \quad y > P.$$

For the positive number $P > 0$, there is a negative number $Q < 0$ such that

$$g(x) > P \quad \text{for every} \quad x < Q.$$

Hence,

$$f(g(x)) < N \quad \text{for every} \quad x < Q.$$

Therefore, $\lim_{x\to-\infty} f(g(x)) = -\infty$.

Q.E.D.

7. Let $A(x) = a$ and $B(x) = b$ be constant functions with $a > 0$ and b any number.

By Exercise 5 (page 99) of Chapter 4, and parts (e) and (f) of Theorem 5.1 (page 101),

$$\lim_{x\to\infty} ax = \infty \quad \text{and} \quad \lim_{x\to-\infty} ax = -\infty.$$

By parts (a) and (b) of Corollary 5.2 (page 106),

$$\lim_{x\to\infty} ax \pm b = \infty \quad \text{and} \quad \lim_{x\to-\infty} ax \pm b = -\infty.$$

8. Let $g(x) = 4x - 1$. By Theorem 4.13 (page 94) and previous Exercise 7,

$$\lim_{x\to-\infty} \frac{1}{(4x-1)\pi} = 0^-.$$

By part (c) of Corollary 5.2 (page 106),

$$\lim_{x\to-\infty} \frac{2}{(4x-1)\pi} = 0.$$

9. Similarly to the previous Exercises 7 and 8,

$$\lim_{x\to-\infty} \frac{1}{(12x-1)\pi} = 0^-.$$

Hence

$$\lim_{x\to-\infty} \frac{6}{(12x-1)\pi} = \lim_{x\to-\infty} 6\left(\frac{1}{(12x-1)\pi}\right) = 0.$$

10. Let $\delta = \dfrac{1}{2\pi} > 0$ be this particular positive number. If x is any number such that $0 < \dfrac{1}{\pi} - x < \delta$, then

$$\begin{aligned}\frac{1}{\pi} - \delta < x < \frac{1}{\pi} &\Rightarrow \frac{1}{2\pi} < x < \frac{1}{\pi}\\ &\Rightarrow \pi < \frac{1}{x} < 2\pi\\ &\Rightarrow 0 < \frac{1}{x} - \pi < 2\pi - \pi = \pi.\end{aligned}$$

Hence, by part (b) of Definition 1.4 (page 4),

$$\frac{1}{x} > \pi \quad \text{for} \quad x \to \frac{1}{\pi}^{-},$$

and by part (d) of Theorem 3.2 (page 47) and Proposition 3.6 (page 61),

$$\lim_{x \to \frac{1}{\pi}^{-}} \frac{1}{x} = \pi^{+}.$$

Since

$$\lim_{x \to \pi^{+}} \sin x = \sin \pi = 0$$

and

$$\sin x < 0 \quad \text{for} \quad x \to \pi^{+},$$

by Proposition 3.6 (page 61)

$$\lim_{x \to \pi^{+}} \sin x = 0^{-}.$$

Therefore, by Theorem 5.4 (page 114)

$$\lim_{x \to \frac{1}{\pi}^{-}} \sin\left(\frac{1}{x}\right) = 0^{-}. \tag{27}$$

Similarly, the function

$$g(x) = x - \frac{1}{\pi} < 0 \quad \text{for} \quad x \to \frac{1}{\pi}^{-};$$

thus,

$$\lim_{x \to \frac{1}{\pi}^{-}} x - \frac{1}{\pi} = 0^{-}. \tag{28}$$

Therefore, the product of these two functions

$$\left(x - \frac{1}{\pi}\right) \sin\left(\frac{1}{x}\right) \quad \text{is positive for } x \to \frac{1}{\pi}^{-},$$

and

$$\lim_{x\to\frac{1}{\pi}^-}\left(x-\frac{1}{\pi}\right)\sin\left(\frac{1}{x}\right)=0^+.$$

By Remark 4.14 (page 96)

$$\lim_{x\to\frac{1}{\pi}^-}\frac{1}{\left(x-\dfrac{1}{\pi}\right)\sin\left(\frac{1}{x}\right)}=\lim_{x\to\frac{1}{\pi}^-}\frac{\csc\left(\frac{1}{x}\right)}{x-\frac{1}{\pi}}=\infty.$$

11. L'Hospital's rule cannot be applied because the function

$$\frac{\csc\left(\dfrac{1}{x}\right)}{x-\dfrac{1}{\pi}}\quad\text{is of type } \infty/0.$$

Indeed, by Remark 4.14 (page 96) and limit (27),

$$\lim_{x\to\frac{1}{\pi}^-}\csc\left(\frac{1}{x}\right)=-\infty,$$

and the limit of the denominator function is zero by (28).

Exercises Chapter VI

Exercises VI (page 154)

1. a. We have that the sine function is continuous on the close interval $\left[0,\dfrac{\pi}{2}\right]$. Hence, the composition

$$\begin{aligned}\sin\circ R\circ S_{-\pi}(x) &= \sin(R(S_{-\pi}(x)))=\sin(-(x-\pi))\\ &= \sin(\pi-x)=\sin x\end{aligned}$$

is continuous in the domain of the composition $\sin\circ R\circ S_{-\pi}(x)$. That is, it is continuous for any

$$0\le\pi-x\le\frac{\pi}{2}\quad\Rightarrow\quad\frac{\pi}{2}\le x\le\pi.$$

We conclude that the sine function is continuous on the close interval $[0, \pi]$. Similarly the composition

$$R \circ \sin \circ S_{-\pi}(x) = -\sin(x - \pi)$$

is continuous for any

$$0 \leq x - \pi \leq \pi \quad \Rightarrow \quad \pi \leq x \leq 2\pi.$$

We conclude that the sine function is continuous on the close interval $[0, 2\pi]$.
The composition

$$\sin(S_{2k\pi}(x)) = \sin(x + 2k\pi)$$

is continuous at $0 \leq x + 2k\pi \leq 2\pi$ for any integer k. Hence,

$$\sin(x + 2k\pi) = \sin x$$

is continuous at $2k\pi \leq x \leq 2(k+1)\pi$ for any integer k. Therefore the sine function is continuous everywhere.

b. Since

$$\sin \circ S_{\pi/2}(x) = \sin\left(\frac{\pi}{2} + x\right) = \cos x \quad \text{for any } x,$$

and the composition of the sine function with any transformation is continuous everywhere, we conclude that the cosine function is also continuous everywhere.

2. Figure B.5 shows the sketch of the composition

$$T_2 \circ R \circ f \circ S_{-2}(x) \quad \text{of Example 6.14 (page 132).}$$

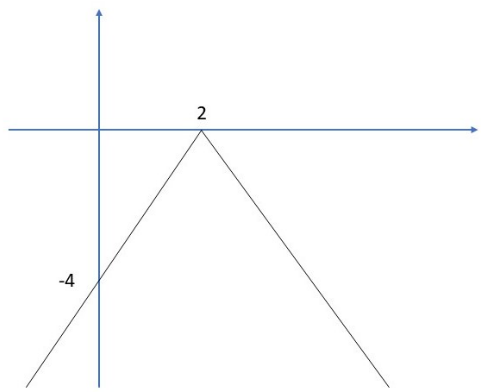


Figure B.5: Sketch of the composition $T_2 \circ R \circ f \circ S_{-2}(x)$

3. Description of the effect that the transformations have on the graph of the function

$$f(x) = \begin{cases} 1 & if \quad x < 0 \\ -1 & if \quad 0 < x \end{cases}$$

under the composition

$$S_2 \circ T_{1/3} \circ R \circ f(x).$$

1. Under the composition

$$R \circ f(x) = -f(x)$$

the graph of the function f is reflected with respect to the x-axis.

2. Under the composition

$$T_{1/3} \circ R \circ f(x) = \frac{-f(x)}{3}$$

the graph of the function $-f(x)$ is compressed vertically by 3 units.

3. Under the composition

$$S_2 \circ T_{1/3} \circ R \circ f(x) = \frac{-f(x)}{3} + 2$$

the graph of the function $\dfrac{-f(x)}{3}$ is shifted 2 units up.

The graph of the function $S_2 \circ T_{1/3} \circ R \circ f(x)$ is shown in Figure B.6.

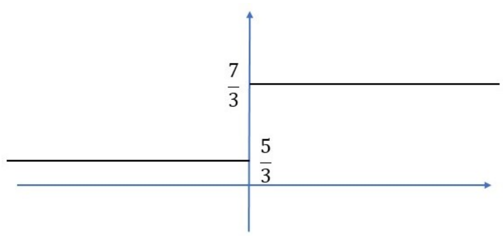


Figure B.6: Sketch of the composition $S_2 \circ T_{1/3} \circ R \circ f(x)$

4. Let f be the piecewise function

$$f(x) = \begin{cases} -1 & if \quad x \leq 0 \\ 1 & if \quad x > 0 \end{cases}$$

a. The domain of f is $\mathbb{R}$ because is defined for any x.

b. It is discontinuous at 0 because

$$\lim_{x \to 0^+} f(x) = 1 \neq -1 = \lim_{x \to 0^-} f(x).$$

c. It is continuous from the left at 0 because

$$\lim_{x \to 0^-} f(x) = -1 = f(0).$$

d. It is continuous on the interval $(0, \infty)$ because for any $a > 0$

$$\lim_{x \to a} f(x) = \lim_{x \to a} 1 = 1 = f(a).$$

5. Let $R(x)$ be the rational function

$$R(x) = \frac{2x^2 - 8x + 6}{3x^2 + 6x + 3}.$$

a. By the statement (6.8) on page 137

$$\lim_{x \to \infty} R(x) = \frac{2}{3}.$$

b. $R(1) = 0 = R(3)$ because $R(x) = \dfrac{2(x-1)(x-3)}{3x^2 + 6x + 3}$.

c. $R(-1)$ is undefined because $R(x) = \dfrac{2x^2 - 8x + 6}{3(x+1)^2}$.

6. We can only determine the type of the limits in parts (a) and (c).

a. $\lim_{x \to 0^+} \dfrac{\ln x}{x}$ type $\infty/0$.

b. $\lim_{x\to\infty} \dfrac{\sin x}{x}$ none because $\lim_{x\to\infty} \sin x$ does not exist.

c. $\lim_{x\to 0^+} \dfrac{\ln x}{\sin x}$ type $\infty/0$.

7. Evaluation of the given limits.

a. The limit

$$\lim_{x\to 0} \frac{1-\sec x}{x} \quad \text{is of type } 0/0.$$

Hence,

$$\lim_{x\to 0} \frac{1-\sec x}{x} = \lim_{x\to 0} \frac{\sec x \tan x}{1} = \sec(0)\tan(0) = 0.$$

b. The limit

$$\lim_{x\to 0} \frac{\tan(ux)}{x} \quad \text{is of type } 0/0.$$

Hence,

$$\lim_{x\to 0} \frac{\tan(ux)}{x} = \lim_{x\to 0} \frac{u\sec^2(ux)}{1} = u\sec^2(0) = u.$$

c. The function

$$\frac{x}{\tan(ux)} \quad \text{is the reciprocal of the function } \frac{\tan(ux)}{x}.$$

Hence, by the Laws of Limits,

$$\lim_{x\to 0} \frac{x}{\tan(ux)} = \frac{1}{u}.$$

8. For any positive number $\delta > 0$ there is a rational u and an irrational v so that $0 < |u|, |v| < \delta$. Hence,

$$\omega(v) \not> 1 \quad \text{for some irrational } 0 < |v| < \delta$$

and

$$\omega(u) \not< -1 \quad \text{for some rational } 0 < |u| < \delta.$$

By Exercise 10 of chapter 3 on page 295

$$\lim_{x \to 0} \omega(x) = 0 \quad \text{from neither the left nor right of zero.}$$

9. If

$$\lim f(x) = 0 \quad \text{and} \quad \lim g(x) = \pm\infty,$$

then by Remark 4.14 (page 96)

$$\lim \frac{1}{g(x)} = 0 \quad \text{from the right or left.}$$

Hence, by the Laws of Limits (Theorem 3.2 on page 47)

$$\lim \frac{f(x)}{g(x)} = \lim f(x) \left(\lim \frac{1}{g(x)} \right) = 0.$$

Exercises Chapter VII

Exercises VII.1 (page 158)

1. Composition of functions.

   a. $F_1(F_4(x)) = F_1\left(\frac{x}{4}\right) = \frac{x}{4} + 4.$

   b. $\sin(C(x)) = \sin(x^3)$

   c. $F_3(S(F_2(x))) = F_3\left(\left(S\left(x - \frac{2}{3}\right)\right)\right) = F_3\left(\left(\left(x - \frac{2}{3}\right)^2\right)\right) = 2\left(x - \frac{2}{3}\right)^2.$

2. Description of the effect that the compositions have on the graph of a function f.

   a. Under the composition

   $$S_3(f(x))) = f(x) + 3$$

   the graph of f is shifted 3 units up.
   Under the composition

   $$T_{1/2}(S_3(f(x))) = \frac{f(x)+3}{2}$$

   the graph of $f(x) + 3$ is shrunk vertically by $1/2$.
   Hence, the graph of f is shifted 3 units up and then shrunk vertically by $1/2$.

   b. Under the composition

   $$f(T_2(x)) = f(2x)$$

   the graph of f is compressed horizontally by a factor of 2 units.

   c. Under the composition

   $$f(R(x) = f(-x)$$

   the graph of f is reflected with respect to the y-axis.
   Under the composition

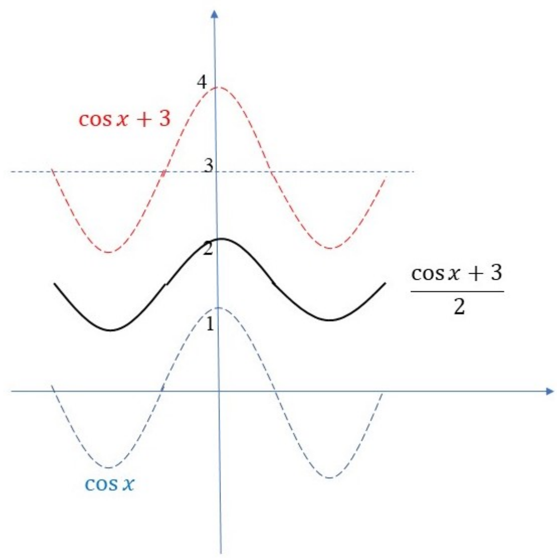
   $$S_{-4}(f(-x)) = f(-x) - 4$$

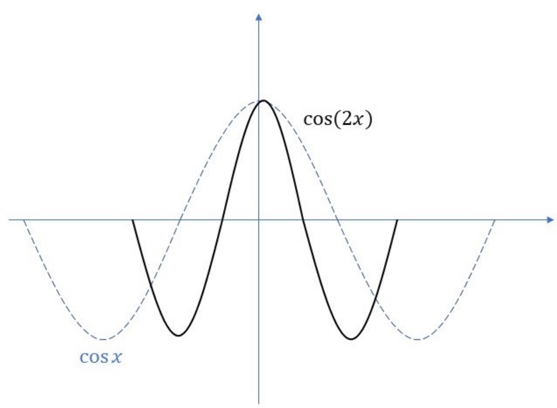
   the graph of $f(-x)$ is shifted 4 units down.
   Hence, the graph of f is reflected with respect to the y-axis and then it is shifted 4 units down.

3. Sketch of the graphs of the functions listed in the previous question where $f(x) = \cos x$.

   a. Shifted 3 units up and then shrunk vertically by $1/2$.

   b. Compressed horizontally by a factor of 2 units.

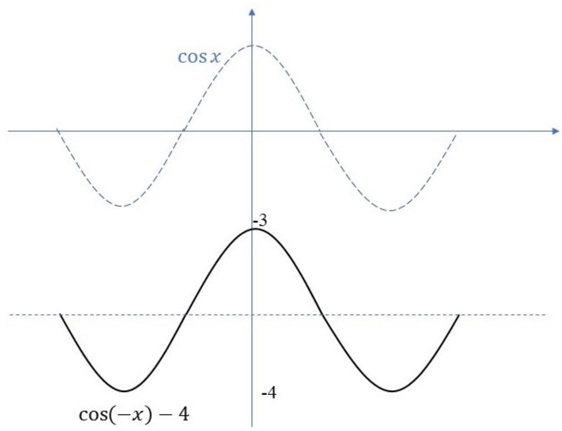




c. Reflected with respect to the y-axis (the cosine is an even function) and then it is shifted 4 units down.

4. The piecewise function defined below is not defined at 0, 2, nor $\frac{5}{2}$.

$$f(x) = \begin{cases} 1 & if \quad x < 0 \\ 1 & if \quad 0 < x < 2 \\ 1 & if \quad 2 < x < \frac{5}{2} \\ 1 & if \quad x > \frac{5}{2} \end{cases}$$



5. For the piecewise function

$$f(x) = \begin{cases} \sin x & \text{if} \quad x < -\frac{\pi}{2} \\ x^2 & \text{if} \quad -\frac{\pi}{2} < x \leq 3 \\ x^3 - 1 & \text{if} \quad \pi \leq x < 2\pi \end{cases}$$

we have

$$f\left(-\frac{\pi}{4}\right) = \left(-\frac{\pi}{4}\right)^2 = \frac{\pi^2}{16},$$

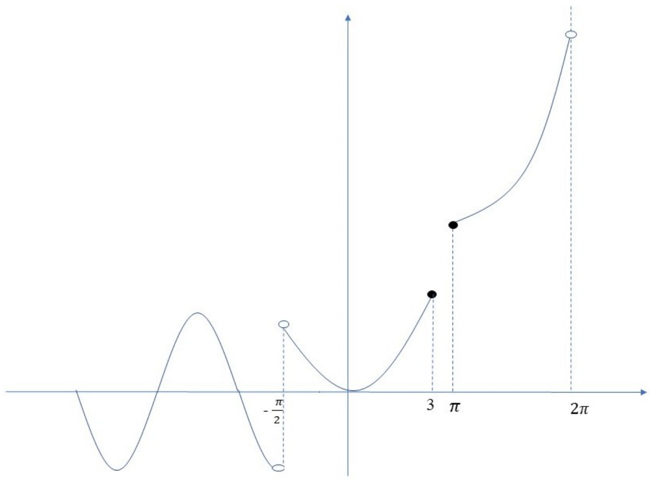
$$f(3) = 3^2 = 9 \quad \text{and} \quad f(7) = 7^3 - 1 = 342.$$

Since $\dfrac{9\pi}{2} > 2\pi$ the number $\dfrac{9\pi}{2}$ is not in the range of f; hence, $f\left(\dfrac{9\pi}{2}\right)$ is undefined.

6. Graph of the function $f(x)$ of Exercise 5.

Exercises VII.2 (page 167)

7. Let $\delta = 0.01$ be a positive number.



a. The number

$$x = \frac{0.01}{2} = \frac{\delta}{2} \quad \text{is positive and less than } \delta.$$

Hence,

$$0 < x < \delta \quad \text{and belongs to the open interval } (0, \delta).$$

b. The distance between the numbers 2 and $2 - \delta$ is

$$|2 - (2 - \delta)| = \delta.$$

Hence, the number

$$x = 2 - \delta + \frac{\delta}{2} = 1.995 \quad \text{belongs to the open interval } (2 - \delta, 2).$$

c. The distance between the numbers 1.001 and $1.001 + \delta$ is

$$|1.001 - (1.001 + \delta)| = \delta.$$

Hence, the number

$$x = 1.001 + \frac{\delta}{2} = 1.006 \quad \text{belongs to the open interval } (1.001, 1.001 + \delta).$$

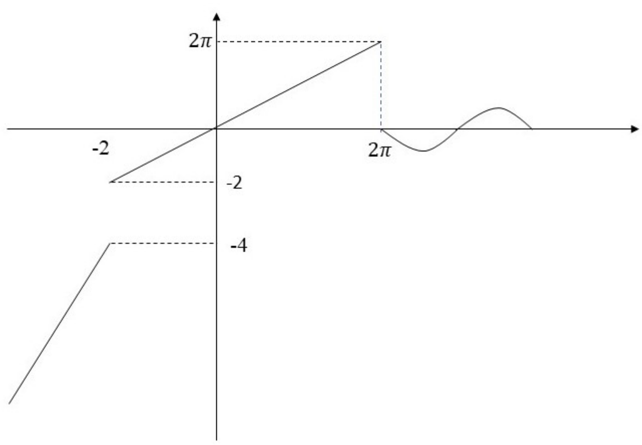
8. The distance between the numbers π and 3.1416 is

$$|\pi - 3.1416| \approx 0.0000073.$$

9. Let g be the piecewise function g.

$$g(x) = \begin{cases} 3x+2 & \text{if} \quad x < -2 \\ x & \text{if} \quad -2 < x < 2\pi \\ 2\sin x & \text{if} \quad x \geq 2\pi \end{cases}$$

A. Graph of the function g



B. Particular values of the function g.

a. $g(-3) = 3(-3) + 2 = -9 + 2 = -7$

b. $g(0) = 0$

c. $g(1) = 1$

d. $g(2\pi) = 2\sin(2\pi) = 0$

e. $g(3\pi) = 2\sin(3\pi) = 0$

C. The number R is:

a. $g(x) \to R = -4$ for $x \to -2^-$.

b. $g(x) \to R = -2$ for $x \to -2^+$.

c. $g(x) \to R = 2\pi$ for $x \to 2\pi^-$.

d. $g(x) \to R = 0$ for $x \to 2\pi^+$.

10. Figure B.7 is the graph of one of infinitely many functions f satisfying all the conditions listed below.

   a. its domain is $\mathbb{R}$,
   b. $f(x) \to -2^+$ as $x \to 2^-$,
   c. $f(x) \to -2^+$ as $x \to 2^+$,
   d. $f(x) \to 0^-$ as $x \to 3^+$, and
   e. $f(x) \to 1$ as $x \to 3^-$.

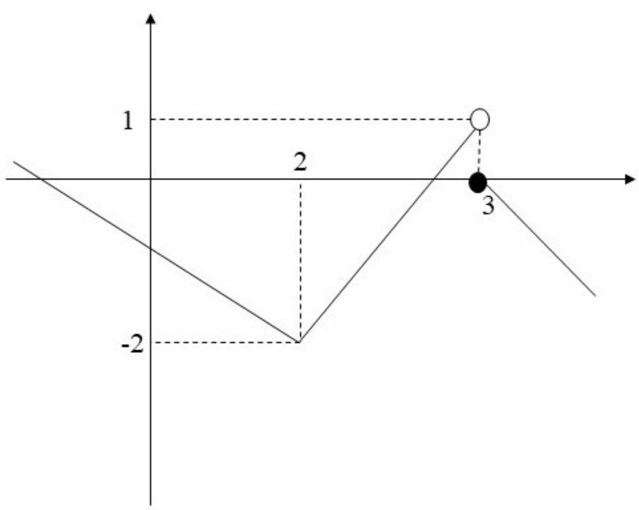


Figure B.7: Graph of a function $f(x)$

Exercises VII.3 (page 174)

11. Figure B.8 is the graph of one of infinitely many functions f satisfying all the conditions (a) - (c).

   a. $\lim_{x \to -1^+} f(x)$ is defined because the function f is defined on the interval $(-1, 0)$.
   b. $\lim_{x \to -1^-} f(x)$ is undefined because the function f is undefined on the interval $(\infty, -1)$.

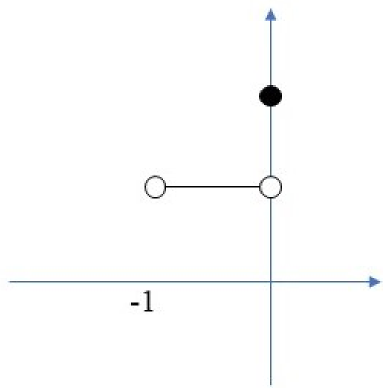


Figure B.8: Graph of a function $f(x)$

c. $f(0)$ is defined and $\lim_{x \to 0} f(x)$ is undefined because the function f is undefined on the interval $(0, \infty)$; hence, $\lim_{x \to 0^+} f(x)$ is undefined.

12. Consider the graph in Figure B.9.

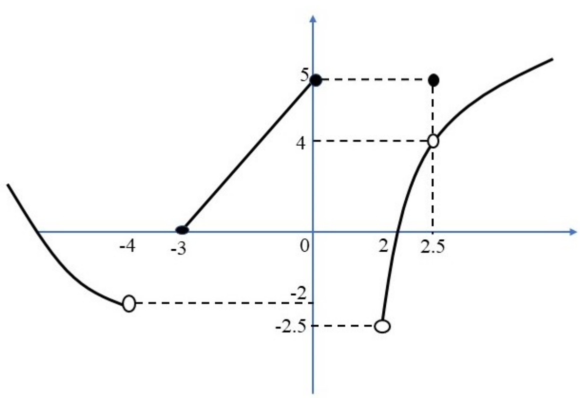


Figure B.9: Graph of the function h

A. $h(a)$ is undefined for any $-4 < a < -3$ or $0 < a < 2$ because the function h is undefined on the intervals $(-4, -3)$ and $(0, 2)$.

B. $h(2.5) = 5$ is defined and $\lim_{x \to 2.5} f(x) = 4 \neq 5 = h(2.5)$.

C. The limit

$$\lim_{x\to 0^+} h(x) \quad \text{is undefined}$$

because the function h is undefined on the interval $(0, 2)$ and therefore on the right of zero.

D. The non-zero number -4 is such that the limit

$$\lim_{x\to -4^+} h(x) \quad \text{is undefined}$$

because the function h is undefined on the interval $(-4, -3)$ and therefore on the right of -4.

E. Value of the given limits.

(a) $\lim_{x\to -3^+} h(x) = 0$

(b) $\lim_{x\to 0^-} h(x) = 5$

(c) $\lim_{x\to 2^+} h(x) = -2.5$

Exercises Chapter VIII

Exercises VIII.1 (page 187)

1. The function $f + g$ is continuous on the domain D_g of g.

2. The domain of the function

$$f(x) = \begin{cases} \sin(x - \pi) & \text{if} \quad x < -\frac{\pi}{2} \\ 2x + \pi & \text{if} \quad -\frac{\pi}{2} \leq x \leq 3\pi \\ -\frac{7x}{2} + \frac{35\pi}{2} & \text{if} \quad x > 3\pi \end{cases}$$

is $\left(-\infty, -\dfrac{\pi}{2}\right) \cup \left(-\dfrac{\pi}{2}, \infty\right)$. Thus,

$$\lim_{x\to 3\pi^-} f(x) = \lim_{x\to 3\pi^-} 2x + \pi = 7\pi = 2(3\pi) + \pi = 7\pi$$

and

$$\lim_{x\to 3\pi^+} f(x) = \lim_{x\to 3\pi^+} -\frac{7x}{2} + \frac{35\pi}{2} = 7\pi.$$

The function f is continuous at 3π, and therefore continuous on its domain.

3. Figure B.10 shows one of infinitely many functions which are continuous on the set $(-\infty, -1) \cup (-1, 3] \cup (3, 6]$ and it is discontinuous at 3.

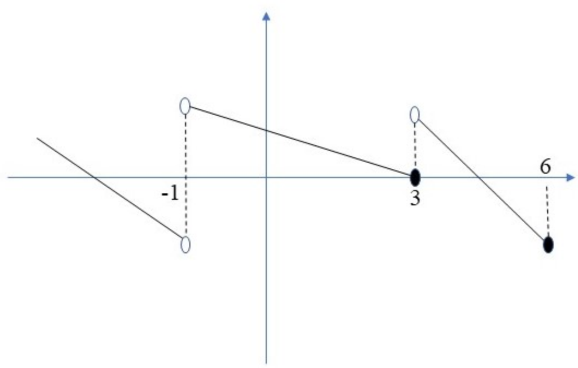


Figure B.10: Graph of the function f

4. The function $f(x) = [x^2 - \sin^3(2x)][3x^2 - \cos(x-3)]$ is the product of everywhere continuous functions.

5. The function $\sin\left(\frac{1}{x}\right)$ is not continuous at zero because its limit at zero does not exist.

6. Figure B.11 shows the graph of one of infinitely many functions which satisfy all the conditions listed below.

    a. f is defined and discontinuous at zero.
    b. $\lim_{x\to 0} f(x) = 1$.
    c. $f(x) > 0$ for all $x > 0$.
    d. f is continuous on the interval $(0, \infty)$.

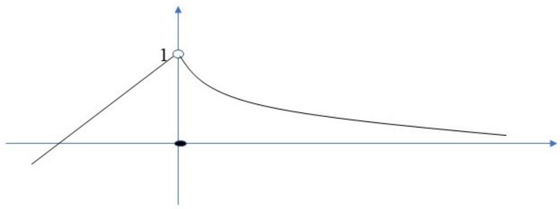


Figure B.11: Graph of the function f

Exercises VIII.2 (page 192)

7. The equality holds for any x except 2 and -2, since

$$\begin{aligned}\frac{x^3+2x^2-4x-8}{x^2-4} &= x+2\\ x^3+2x^2-4x-8 &= (x^2-4)(x+2)\\ x^3+2x^2-4x-8 &= x^3+2x^2-4x-8\end{aligned}$$

and the quotient

$$\frac{x^3+2x^2-4x-8}{x^2-4} \quad \text{is not defined at 2 and } -2.$$

8. The domain of the function

$$f(x) = \frac{x^3+2x^2-4x-8}{x^2-4} \quad \text{is all real numbers except 2 and } -2$$

and the domain of the function

$$g(x) = x+2 \quad \text{is all real numbers.}$$

Hence they are not equal.

9. Figure B.12 shows the graph of the linear function g.

Figure B.13 shows the graph of the linear function f undefined at 2 and -2.

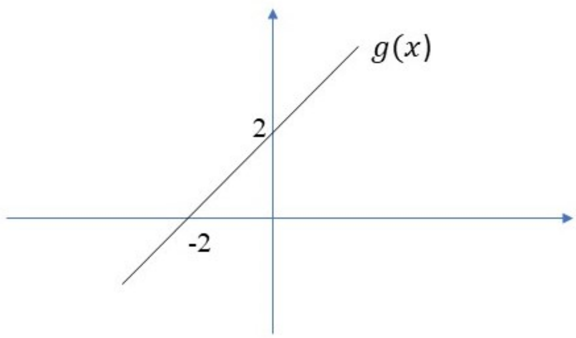


Figure B.12: Graph of the function g of Exercise 2

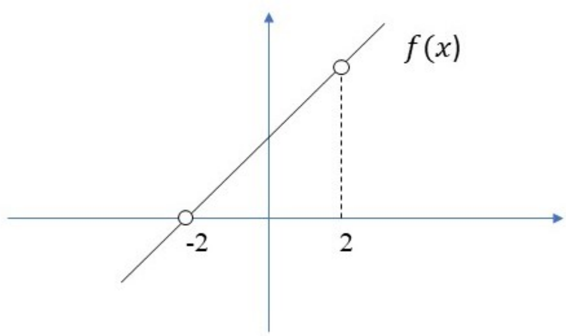


Figure B.13: Graph of the function f of Exercise 2

10. Evaluate the given limits by simplifying the given function.

    a. Applying the identity $1 + \tan^2 x = \sec^2 x$

    $$\frac{1 - \sec^2(x-1)}{x-1} = \frac{-\tan^2(x-1)}{x-1}.$$

    If $y = x - 1$, then $y \to 0$ as $x \to 1$.
    Thus, by the laws of limits

    $$\begin{aligned} \lim_{x \to 1} \frac{-\tan^2(x-1)}{x-1} &= \lim_{y \to 0} -\left(\frac{\sin y}{y}\right) \frac{\sin y}{\cos^2 y} \\ &= \left(\lim_{y \to 0} -\left(\frac{\sin y}{y}\right)\right) \left(\lim_{y \to 0} \frac{\sin y}{\cos^2 y}\right) = 0 \end{aligned}$$

b. By the properties of natural logarithm

$$\ln(x-1)+\ln(x+1)-\ln(2x^2-2)=\ln\left(\frac{(x-1)(x+1)}{2(x^2-1)}\right)$$
$$=\ln\left(\frac{x^2-1}{2(x^2-1)}\right)=-\ln 2.$$

Hence,

$$\lim_{x\to 1}\ln(x-1)+\ln(x+1)-\ln(2x^2-2)=-\ln 2.$$

c. By simplifying, for any $x\neq 3$

$$\frac{x^3-3x^2+2x-6}{x-3}=\frac{(x^2+2)(x-3)}{x-3}=x^2+2.$$

Hence,

$$\lim_{x\to 3}\frac{x^3-3x^2+2x-6}{x-3}=\lim_{x\to 3}x^2+2=11.$$

Exercises VIII.3 (page 197)

11. Figure B.14 shows the graph of one of infinitely many functions whose limit at the number a is L from the left. **??**

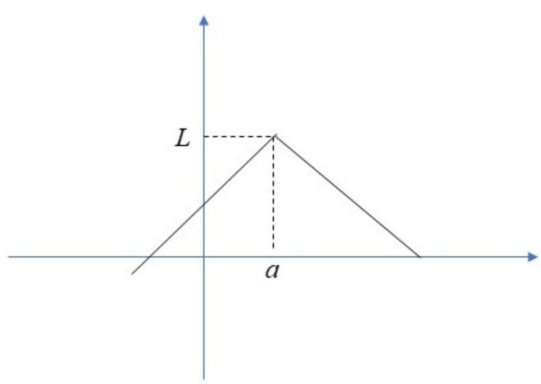


Figure B.14: Limit at a is L

12. Figure B.15 shows the graph of one of infinitely many functions whose limit at the number a from the right is L from the right.

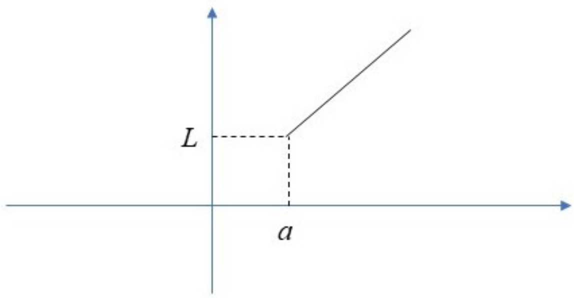


Figure B.15: Limit at a from the right is L

13. Evaluate the limits listed below and determine whether the limits are from the right or left.

    a. The rational function

    $$f(x) = \frac{x+2}{x^2+4x+4} \quad \text{is continuous at } 1.$$

    Hence,

    $$\lim_{x \to 1} \frac{x+2}{x^2+4x+4} = \frac{1}{3}.$$

    The limit is $\frac{1}{3}$ from the right, since

    $$f(x) = \frac{x+2}{x^2+4x+4} > 0 \quad \text{for any } 0 < x < 2.$$

    b. The polynomial function

    $$f(x) = x^3 + 3x^2 - 2x - 6 \quad \text{is continuous at } 2;$$

    hence,

    $$\lim_{x \to 2^-} x^3 + 3x^2 - 2x - 6 = 10.$$

    The limit is 10 from the right, since

    $$x^3 + 3x^2 - 2x - 6 = (x+3)(x^2-2) > 0 \quad \text{for any } \sqrt{2} < x < 2.$$

c. The function $f(x) = x\sin(2x)$ is continuous everywhere; hence,

$$\lim_{x\to\pi/2^+} x\sin(2x) = \left(\frac{\pi}{2}\right)\sin\left(\frac{2\pi}{2}\right) = 0.$$

The limit is zero from the left, since

$$x\sin(2x) < 0 \quad \text{for any } \frac{\pi}{2} < x < \frac{3\pi}{4}.$$

Exercises Chapter IX

Exercises IX.1 (page 207)

1. The rational function

$$\frac{1}{2x^2 - x - 6} = \frac{1}{(2x+3)(x-2)}$$

is continuous for every x except 2 and 3/2.

2. Since $x^2 + 1 > 0$ for any x, the function

$$f(x) = \sqrt{x^2+1} \quad \text{is defined on } \mathbb{R},$$

and therefore

$$\frac{1}{x^2+1} \quad \text{is continuous everywhere.}$$

3. Since

$$x^3 - 3 = (x - \sqrt[3]{3})(x^2 + \sqrt[3]{3}x + \sqrt[3]{9}),$$

the rational function

$$\frac{1}{x^3-3} \quad \text{is undefined at } \sqrt[3]{3}$$

and therefore is not continuous everywhere.

4. Let $a = 10^{11}$ and $b = 10^{13}$. Thus,

$$a < 10^{12} < b \quad \text{since } a(10) = 10^{13} \text{ and } 10^{12}(10) = b.$$

Note. There are infinitely many such numbers a and b.

5. Let $a = -10^{11}$ and $b = -10^{13}$. By Exercise 4, and inequality 6 (page xii)

$$10^{11} < 10^{12} < 10^{13} \quad \Rightarrow \quad \frac{1}{10^{13}} < \frac{1}{10^{12}} < \frac{1}{10^{11}}.$$

By inequality 5 (page xii)

$$\frac{1}{a} = -\frac{1}{10^{11}} < -\frac{1}{10^{12}} < -\frac{1}{10^{13}} = \frac{1}{b}.$$

Note. There are infinitely many such numbers a and b.

6. Reciprocal of the given numbers.

   a. Reciprocal of $\frac{3}{8}$ is $\frac{8}{3}$.

   b. Reciprocal of -5 is $-\frac{1}{5}$

   c. Reciprocal of $\frac{5-7}{\sqrt{2}+2}$ is $-\frac{\sqrt{2}+2}{2} = -\frac{\sqrt{2}}{2} - 1$.

   d. Reciprocal of $3^{-6} = \frac{1}{3^6}$ is 3^6.

7. The relation in increasing order of the given numbers is:

$$-\sqrt{65} < -1 < 1 - \sqrt{3} < 3 < \pi < \frac{1}{\sqrt{2\pi}}.$$

8. Let $n = \frac{4}{5}$. Thus,

$$\frac{3}{5} < \frac{4}{5} < \frac{5}{5} = 1.$$

Note. There are infinitely many such numbers n.

9. Let $n = 5$. Thus,

$$\frac{8}{15} < \frac{6}{n} \quad \Rightarrow \quad 8(5) = 40 < 6(15) \quad \Rightarrow \quad \frac{8}{15} < \frac{6}{5}.$$

Note. There are infinitely many such numbers n.

10. We have that

$$\frac{n}{6} < \frac{7}{n} < \frac{5n}{9} \quad \Rightarrow \quad n^2 < 6(7) = 42 \quad \text{and} \quad 7(9) = 63 < 5n^2.$$

Since $n^2 < 42$ for $1 \leq n \leq 6$ and $63 < 5n^2$ for $n \geq 4$, both inequalities holds for $4, 5$ and 6 only.

Exercises IX.2 (page 218)

11. Let $V > 0$ be any positive number. For this positive number there is

$$u = \frac{\ln((V+1)^3)}{2}$$

so that the number $e^{2u/3}$ belongs to the set S and

$$e^{2u/3} = \exp\left(\frac{2\ln((V+1)^3)}{2(3)}\right) = \exp\left(\frac{3\ln(V+1)}{3}\right) = V+1 > V.$$

By Definition 9.9 the set S is unbounded above.

12. a. Negation of Definition 9.9.
A set S is bounded above if there is a positive number $B_A > 0$ such that

$$x \leq B_A \quad \text{for every } x \in S.$$

b. Negation of Definition 9.10.
A set S is bounded below if there is a negative number $B_B < 0$ such that

$$x \geq B_B \quad \text{for every } x \in S.$$

c. A set S is bounded if its is bounded above and below. Hence, S is bounded if there is a positive number $B > 0$ such that

$$|x| \leq |B| \quad \text{for every } x \in S.$$

13. The function $f(x) = \dfrac{1}{x^2}$ is the reciprocal of the basic square function $S(x) = x^2$.

    1. The function S is continuous and positive on the intervals $(-\infty, 0)$ and $(0, \infty)$. Hence, the function f is continuous and positive on intervals $(-\infty, 0)$ and $(0, \infty)$.
    2. The function S never negative. Hence, the function f is never negative.
    3. The function S is zero at zero only. Hence, the function f is undefined at zero.
    4. The function S is very large on the intervals $(-\infty, -1)$ and $(1, \infty)$. Hence the function f is very small on the intervals $(-\infty, -1)$ and $(1, \infty)$.
    5. The function S is positive and close to zero on the intervals $(-1, 0)$ and $(0, 1)$. Hence, the function f is positive and very large on the intervals $(-1, 0)$ and $(0, 1)$.
    6. Step 6 does not apply.
    7. Step 7 does not apply.
    8. Figure B.16 shows the sketch of graph of the function $f(x) = \dfrac{1}{x^2}$.

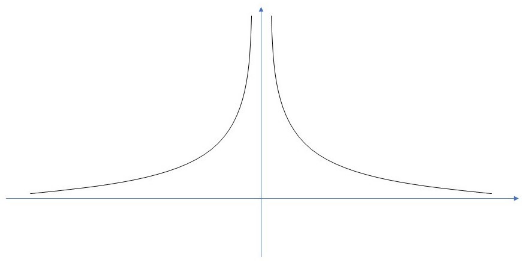


Figure B.16: Graph of the function $f(x) = \dfrac{1}{x^2}$

14. Figure B.17 shows the sketch of the graph of the function $\dfrac{1}{f(x)}$ where

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \neq 0 \\ 2 & \text{if} \quad x = 0 \end{cases}$$

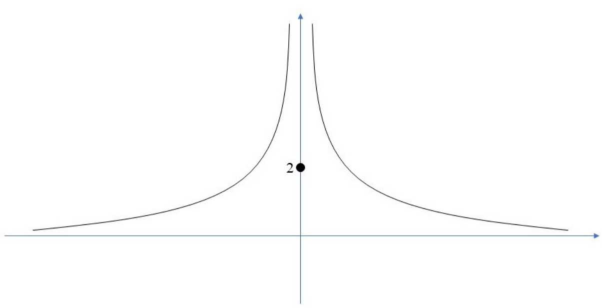


Figure B.17: Sketch of the function $\dfrac{1}{f(x)}$

15. Since

$$\frac{1}{I(x)} < 0 \quad \text{for } x \to 0^-,$$

the given set is bounded above with bound 1.
Let $\delta > 0$ be a positive number, so that

$$x \to 0^- = \{x \in \mathbb{R} \mid -\delta < x < 0\}.$$

For any negative number $U < 0$ the number

$$N = \max\left(\frac{1}{U-1}, -\delta\right),$$

is such that any number $x > N$ belongs to the set $x \to 0^-$ and therefore
$\frac{1}{x}$ belongs to the given set. Moreover,

$$x > \frac{1}{U-1} \quad \Rightarrow \quad \frac{1}{x} < U - 1 < U.$$

And the given set is unbounded below.

Exercises IX.3 (page 230)

16. Functions which are unbounded (below or above) on the indicated range.

    a. The function $f(x) = x \sin x$ for all $x \in \mathbb{R}$ is unbounded. Since,

    $$\lim_{x\to\infty} \frac{2\pi x}{2} = \infty \quad \text{and} \quad \lim_{x\to-\infty} \frac{2\pi x}{2} = -\infty,$$

    for any nonzero number B there are integers k, n such that

    $$\frac{2k\pi}{2} > |B| \quad \text{and} \quad \frac{2n\pi}{2} < -|B|.$$

    Hence, for $u = \dfrac{2k\pi x}{2}$ and $v = \dfrac{2n\pi}{2}$

    $$u \sin u = \frac{2k\pi}{2} \sin\left(\frac{2k\pi}{2}\right) = \frac{2k\pi}{2} > |B|$$

    and

    $$v \sin v = \frac{2n\pi}{2} \sin\left(\frac{2n\pi}{2}\right) = \frac{2n\pi}{2} < -|B|.$$

    Therefore, the function f is neither bounded below nor above.

b. The function

$$g(x) = \tan x \quad \text{for } 0 < x < \frac{\pi}{2}$$

is bounded below with bound zero, since,

$$\tan x \geq 0 \quad \text{for } 0 < x < \frac{\pi}{2}.$$

It is unbounded above, because for any positive number B, there is

$$0 < y = \tan^{-1}(2B) < \frac{\pi}{2}$$

so that

$$\tan y = \tan(\tan^{-1}(2B)) = 2B > B.$$

c. The function

$$h(x) = \ln x \quad \text{for} \quad 0 < x < 1$$

is bounded above with bound zero, since

$$\ln x < 0 \quad \text{for } 0 < x < 1.$$

It is unbounded below, because for any negative number B, there is $0 < u = e^{B-1} < 1$ so that

$$\ln u = \ln(e^{B-1}) = B - 1 < B.$$

17. The function

$$f(x) = \frac{1}{x} \qquad \text{is one of such infinitely many functions.}$$

This function is bounded below with bound 0, since

$$f(x) \geq 0 \quad \text{for every } 0 < x < 1.$$

For any number $B > 0$, there is $0 < u = \dfrac{1}{B+1} < 1$ such that

$$\frac{1}{u} = B + 1 > B.$$

Hence f is unbounded above.

18. We apply Definition 8.5 and Theorem 9.17 in the evaluation of the given limits.

    a. The polynomial function $x^2 - 4$ is continuous everywhere and $x^2 - 4 > 0$ for $x > 2$; hence,

    $$\lim_{x \to 2^+} x^2 - 4 = 2^2 - 4 = 0^+.$$

    By Theorem 9.17

    $$\lim_{x \to 2^+} \frac{1}{x^2 - 4} = \infty.$$

    b. The polynomial function $x^2 - 4$ is continuous everywhere and $x^2 - 4 < 0$ for $x < 2$; hence,

    $$\lim_{x \to 2^-} x^2 - 4 = 2^2 - 4 = 0^-.$$

    By Theorem 9.17

    $$\lim_{x \to 2^-} \frac{1}{x^2 - 4} = -\infty.$$

    c. The cosine function is continuous everywhere and $\cos x < 0$ for $\pi < x < 3\pi/2$; hence,

    $$\lim_{x \to \pi^+} \cos x = \cos(\pi) = -1^-.$$

    By Theorem 9.17

    $$\lim_{x \to \pi^+} \sec x = -\infty.$$

    d. The function $\ln(x-1)$ is continuous on $(1, \infty)$ and $\ln(x-1) > 0$ for $x > 2$; hence,

    $$\lim_{x \to 2^+} \ln(x-1) = \ln(2-1) = 0^+.$$

    By Theorem 9.17

    $$\lim_{x \to 2^+} \frac{1}{\ln(x-1)} = \infty.$$

19. Vertical asymptotes of the given functions.

a The only possible asymptote of the function

$$f(x) = \frac{1}{x^3 - 8} \quad \text{is } x = 2.$$

Since $x^3 - 8$ is a polynomial and

$$x^3 - 8 = (x-2)(x^2 + 2x + 4) > 0 \quad \text{for } x > 2$$

we conclude that

$$\lim_{x \to 2^+} x^3 - 8 = 2^3 - 8 = 0^+.$$

By Theorem 9.17

$$\lim_{x \to 2^+} \frac{1}{x^3 - 8} = \infty.$$

Indeed, the vertical line $x = 2$ is a vertical asymptote.

b The only possible vertical asymptotes are $x = \sqrt{2}$ and $x = -\sqrt{2}$.
If $t = x^2 - 1$,

$$\lim_{x \to \sqrt{2}+} x^2 - 1 = 1^+ \quad \Rightarrow \quad \lim_{t \to 1^+} \ln t = 0^+.$$

By Theorem 9.17

$$\lim_{x \to \sqrt{2}^+} \frac{1}{\ln(x^2 - 1)} = \infty.$$

Also,

$$\lim_{x \to -\sqrt{2}-} x^2 - 1 = 1^+ \quad \Rightarrow \quad \lim_{t \to 1^+} \ln t = 0^+.$$

By Theorem 9.17

$$\lim_{x \to -\sqrt{2}^-} \frac{1}{\ln(x^2 - 1)} = \infty.$$

Therefore the vertical lines $x = \sqrt{2}$ and $x = -\sqrt{2}$ are vertical asymptotes of the function g.

c The possible vertical asymptotes are $x = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$.
If $t = x - \pi$,

$$\lim_{x \to \pi/2^+} x - \pi = -\frac{\pi}{2}^+ \quad \Rightarrow \quad \lim_{t \to -\pi/2^+} \cos t = 0^+.$$

By Theorem 9.17

$$\lim_{x \to \pi/2^+} \sec(x - \pi) = \infty.$$

Also, if $t = \dfrac{3\pi}{2}$,

$$\lim_{x \to (3\pi/2)^+} x - \pi = \frac{\pi}{2}^+ \quad \Rightarrow \quad \lim_{t \to \pi/2^+} \cos t = 0^-.$$

By Theorem 9.17

$$\lim_{x \to (3\pi/2)^+} \sec(x - \pi) = -\infty.$$

Therefore the vertical lines $x = \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ are vertical asymptotes of the function h on the interval $[0, 2\pi]$.

20. Graphs of the given functions.

    a Figure B.18 shows the graph of the natural logarithm function shifted one unit to the right.

    b Figure B.19 shows the graph of the reciprocal function of the function $\ln(x - 1)$.

21. The application of Theorem 9.17 (page 224) shown below is incorrect because the Laws of Limits cannot be applied in step

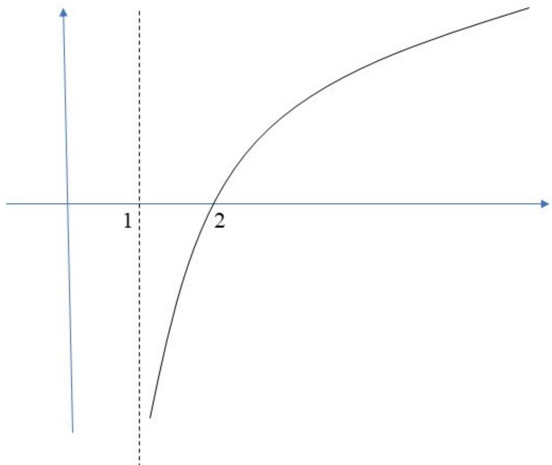


Figure B.18: Sketch of the function $f(x) = \ln(x-1)$

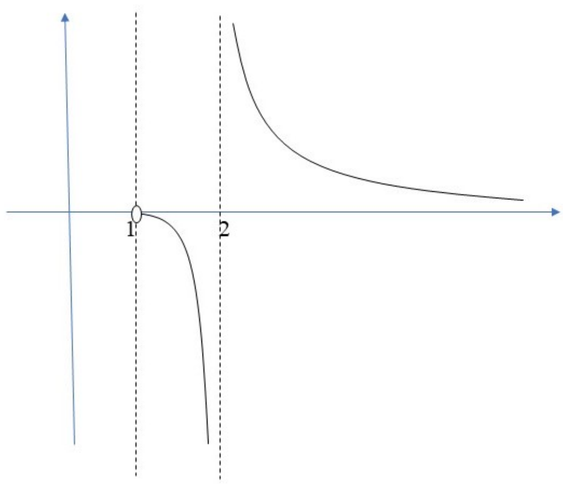


Figure B.19: Sketch of the function $g(x) = \dfrac{1}{\ln(x-1)}$

(21a) since both limits are not finite.

$$\lim_{x\to 1^+} \frac{1}{(x-1)\sin(\pi x)} = \lim_{x\to 1^+} \frac{1}{(x-1)} \frac{1}{\sin(\pi x)}$$

$$= \lim_{x\to 1^+} \frac{1}{(x-1)} \lim_{x\to 1^+} \frac{1}{\sin(\pi x)} \tag{21a}$$

$$= \lim_{x\to 1^+} \frac{1}{0^+} \lim_{x\to 1^+} \frac{1}{0^-} \tag{21b}$$

$$= (\infty)(-\infty) = -\infty. \tag{21c}$$

Moreover, the step (21b) is incorrect because both expressions $\frac{1}{0^+}$ and $\frac{1}{0^-}$ are undefined, and the multiplication (21c) is also undefined.

The correct application of Theorem 9.17 is as follows.

The function $(x-1)\sin(\pi x)$ is everywhere continuous; hence,

$$\lim_{x\to 1^+}(x-1)\sin(\pi x) = (1-1)\sin(\pi) = 0.$$

Since $(x-1)\sin(\pi x) < 0$ for $1 < x < 3/2$, by Theorem 9.17

$$\lim_{x\to 1^+}\frac{1}{(x-1)\sin(\pi x)} = -\infty.$$

Exercises IX.4 (page 236)

22. Since $\ln x < 0$ for $0 < x < 1$, by Theorem 9.17

$$\lim_{x\to 1^-}\frac{1}{\ln x} = -\infty.$$

The cosine function is everywhere continuous; hence,

$$\lim_{x\to 1^-}\cos x = \cos 1 > 0.$$

Therefore, by Theorem 9.18

$$\lim_{x\to 1^-}\frac{\cos x}{\ln x} = -\infty.$$

23. The possible vertical asymptotes of the function

$$\frac{x^3-1}{(x^2-1)(x+3)} \quad \text{are } x = \pm 1 \text{ and } x = -3.$$

Factoring and simplifying

$$\frac{x^3-1}{(x^2-1)(x+3)} = \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)(x+3)} = \frac{x^2+x+1}{(x+1)(x+3)}.$$

for any $x \neq \pm 1, -3$. The vertical line $x = 1$ is not a vertical asymptote.
The polynomial function $x^2 + x + 1$ is continuous everywhere; hence,

$$\lim_{x\to -1} x^2+x+1 = (-1)^2-1+1 = 1 \quad \text{and} \quad \lim_{x\to -3} x^2+x+1 = (-3)^2-3+1 = 7.$$

Since $(x+1)(x+3) > 0$ for $-1 < x < 0$ and $-4 < x < -3$,

$$\lim_{x\to -1^+} \frac{1}{(x+1)(x+3)} = \infty = \lim_{x\to -3^-} \frac{1}{(x+1)(x+3)}.$$

By Theorem 9.18

$$\lim_{x\to -1^+} \frac{x^3-1}{(x^2-1)(x+3)} = \infty = \lim_{x\to -3^-} \frac{x^3-1}{(x^2-1)(x+3)}.$$

Therefore the vertical lines $x = -1$ and $x = -3$ are vertical asymptotes.

24. The given application of Theorem 9.18 is incorrect because

$$\begin{aligned} \lim_{x\to\pi^-} \frac{3x}{\cos(3x)} &= \lim_{x\to\pi^-} \frac{3\pi}{\cos(3\pi)} && \cos(3\pi) \text{ is undefined} \\ &= \lim_{x\to\pi^-} \frac{3\pi}{0^+} && \text{division by zero is undefines} \\ &= 3(\infty) = \infty && 3(\infty) \text{ is undefined.} \end{aligned}$$

Moreover, a limit is not equal to a limit of a constant. The correct application is as follows.

The functions $3x$ and $\cos(3x)$ are continuous everywhere; hence,

$$\lim_{x\to\pi^-} 3x = 2\pi > 0 \quad \text{and} \quad \lim_{x\to\pi^-} \cos(3x) = \cos(3\pi) = 0.$$

Since $\cos(3x) > 0$ for $x \to \pi^-$, by Theorem 9.17

$$\lim_{x\to\pi^-} \frac{1}{\cos(3x)} = \infty.$$

Therefore, by Theorem 9.18

$$\lim_{x \to \pi^-} \frac{3x}{\cos(3x)} = \infty.$$

25. Let $f(x) = \cos x$ and $g(x) = x^2$. Then,

$$\lim_{x \to 0} -\cos x = -1 \quad \text{and} \quad \lim_{x \to 0} x^2 = 0.$$

Since $x^2 > 0$ for any nonzero x, by Theorem 9.17 and Theorem 9.18

$$\lim_{x \to 0} \frac{-\cos x}{x^2} = -\infty.$$

Exercises IX.5 (page 239)

26. The function $f(x) = x^2 \sin x$ does not have vertical asymptotes on its domain $\mathbb{R}$ because it is continuous everywhere.

27. Let $f(x) = e^x$ be an unbounded function and let $g(x) = \sin x$ be a bounded function. Hence,

$$f(g(x)) = \exp(\sin x) \quad \text{is bounded.}$$

Indeed, since $-1 \leq \sin x \leq 1$ for any x, and the exponential function is increasing and continuous,

$$e^{-1} \leq \exp(\sin x) \leq e \quad \text{for every } x.$$

The lower bound of $f \circ g$ is e^1 and its upper bound is e.

28. Since the rational function $\dfrac{\pi x}{x+2}$ is continuous at 2, we have

$$\lim_{x \to 2} \frac{\pi x}{x+2} = \frac{\pi}{2}.$$

Moreover, by Theorem 9.18

$$\lim_{x \to 2} \frac{\pi x}{x+2} = \frac{\pi^+}{2} \quad \text{because} \quad \frac{\pi x}{x+2} > 0 \text{ for } x \to 2.$$

Hence, by Theorem 9.19

$$\lim_{x \to 2} \sec\left(\frac{\pi x}{x+2}\right) = \lim_{t \to \pi/2^+} \sec t = -\infty.$$

Therefore, the vertical line $x = \dfrac{\pi}{2}$ is a vertical asymptote of the function $f(x) = \sec\left(\dfrac{\pi x}{x+2}\right)$.

29. Let $t = x^2 - 1$ be a quadratic function. Since $t \to 0^+$ for $x \to 1^+$, by Theorem 9.19

$$\lim_{x \to 1^+} \ln(x^2 - 1) = \lim_{t \to 0^+} \ln t = -\infty.$$

30. The evaluation of the given limit is incorrect because

$$\begin{aligned} \lim_{x \to 1^+} \ln(-\sin(\pi x)) &= \ln(-\sin(\pi)) && \ln(-\sin(\pi)) \text{ is undefined} \\ &= \ln(0^+) && \ln(0^+) \text{ is undefined} \\ &= -\infty && \text{the equality is incorrect.} \end{aligned}$$

Its correct evaluation is as follows.

The composition

$$-\sin(\pi x) \quad \text{is continuous everywhere}$$

and

$$\pi x \to \pi^+ \quad \text{for } x \to 1^+.$$

Hence,

$$\lim_{x \to 1^+} -\sin(\pi x) = \lim_{t \to \pi^+} -\sin t = 0^+.$$

By Theorem 9.19 (page 236)

$$\lim_{x \to 1^+} \ln(-\sin(\pi x)) = \lim_{t \to 0^+} \ln t = -\infty.$$

Exercises Chapter X

Exercises X.1 (page 242)

1. Figure B.20 shows the graph of a function which is discontinuous at all the negative integers and it has $y = 0$ as a horizontal asymptote in the negative direction.

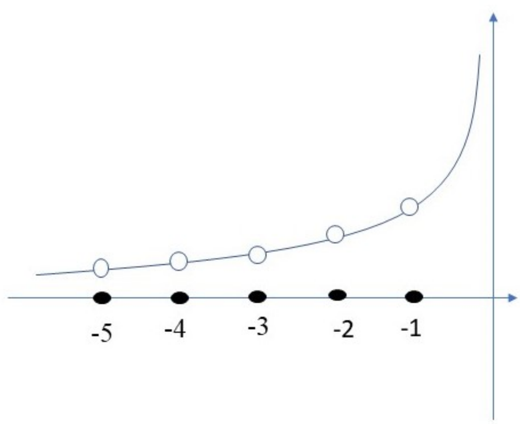


Figure B.20: Sketch of a discontinuous function with a horizontal asymptote

2. Figure B.21 shows the graph of an everywhere continuous function with horizontal asymptotes $y = 1$ and $y = -1$.

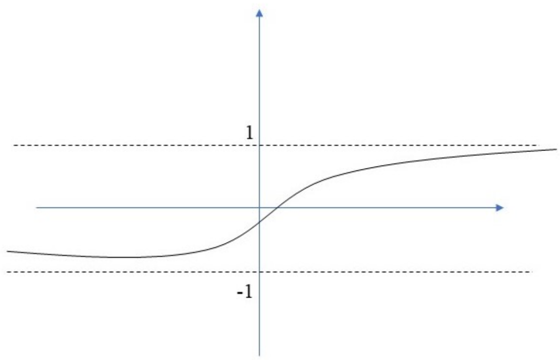


Figure B.21: Graph of a continuous function with two horizontal asymptote

3. Figure B.22 shows the graph of a non-constant, everywhere continuous function with the horizontal asymptote $y = 2$ in both directions positive and negative.

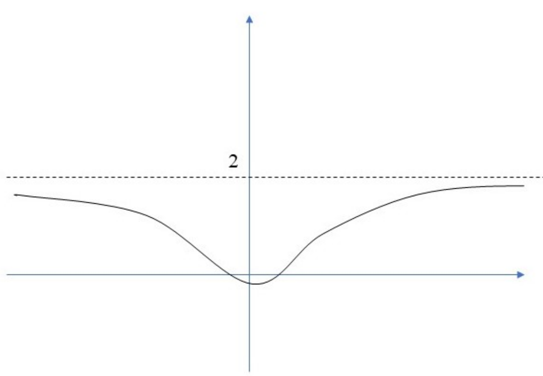


Figure B.22: Graph of a continuous function with one horizontal asymptote

4. The sine function does not have a horizontal asymptote because its limits at infinity and negative infinity do not exist.

Exercises X.2 (page 249)

5. Figure B.23 shows the graph of a function whose limit at negative infinity is L from the right.

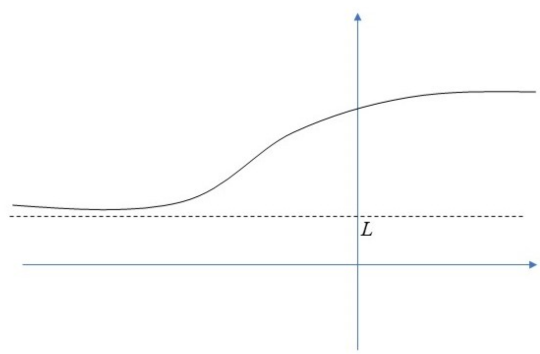


Figure B.23: Graph of a function with horizontal asymptote $y = L$ from the right

6. The limit

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

is neither zero from the right nor the left because as shown in Example 2.5 (page 33), for any $M > 0$, there is a number $u > M$ such that

$$\sin u = 0.$$

Hence, for $x \to \infty$

$$\frac{\sin u}{u} \not< 0 \quad \text{and} \quad \frac{\sin v}{v} =\not> 0.$$

7. Evaluation of the given limits.

a. By (10.2) on page 245

$$\lim_{x\to\infty} \frac{3x^2-1}{x^6+2x-1} = 0.$$

Since,

$$\frac{3x^2-1}{x^6+2x-1} > 0 \quad \text{for any } x > 1,$$

by Definition 10.1 (page 245)

$$\lim_{x\to\infty} \frac{3x^2-1}{x^6+2x-1} = 0^+.$$

b. By (10.2) on page 245 and the Laws of Limits

$$\lim_{x\to\infty} \frac{2x^2-x}{2x^2+2} - \frac{3x^2-7x^3}{x^3+2x^2+2} = \lim_{x\to\infty} \frac{2x^2-x}{2x^2+2} - \lim_{x\to\infty} \frac{3x^2-7x^3}{x^3+2x^2+2} = 1+7 = 8.$$

Since

$$\frac{2x^2-x}{2x^2+2} - \frac{3x^2-7x^3}{x^3+2x^2+2} = \frac{2x^2-x}{2x^2+2} + \frac{7x^3+3x^2}{x^3+2x^2+2} > 0 \quad \text{for any } x > 1,$$

by Definition 10.1 (page 245)

$$\lim_{x\to\infty} \frac{2x^2-x}{2x^2+2} - \frac{3x^2-7x^3}{x^3+2x^2+2} = 8^+.$$

c. By (10.2) on page 245 and the Laws of Limits

$$\lim_{x\to-\infty} \frac{x^4+3x^2-7}{x^6+x^3} + \frac{x^2-3x}{4-5x^2} = \lim_{x\to-\infty} \frac{x^4+3x^2-7}{x^6+x^3} + \lim_{x\to-\infty} \frac{x^2-3x}{4-5x^2} = 0 - \frac{1}{5}.$$

Since,

$$f(x) = \frac{x^4 + 3x^2 - 7}{x^6 + x^3} > 0 \quad \text{for any } x,$$

by Definition 10.1

$$\lim_{x \to -\infty} \frac{x^4 + 3x^2 - 7}{x^6 + x^3} = 0^+.$$

Since,

$$g(x) = \frac{x^2 - 3x}{4 - 5x^2} < 0 \quad \text{for any } x < -2,$$

again by Definition 10.1

$$\lim_{x \to -\infty} \frac{x^2 - 3x}{4 - 5x^2} = -\frac{1}{5}^-.$$

Hence, there exist $V_1, V_2 < 0$ such that

$$0 < f(x) < \frac{2}{5} \quad \text{for any } x < V_1$$

and

$$g(x) < -\frac{1}{5} \quad \text{for any } x < V_2.$$

Let V be the negative number $V = \min(V_1, V_2)$.
For $x < V$ both statements above hold; thus, for any $x < V$

$$0 < f(x) < \frac{2}{5} < \frac{1}{5} < -g(x) \quad \Rightarrow$$
$$f(x) + g(x) < \frac{2}{5} + g(x) < \frac{2}{5} - \frac{1}{5} = \frac{1}{5}.$$

By Definition 10.1

$$\lim_{x \to -\infty} \frac{x^4 + 3x^2 - 7}{x^6 + x^3} + \frac{x^2 - 3x}{4 - 5x^2} = \frac{1}{5}^-.$$

8. Evaluation of the given limits.

   a. If $t = \csc x$, then $t \to -\infty$ as $x \to 0^-$. Thus,

   $$\lim_{x \to 0^-} \cot^{-1}(\csc x) = \lim_{t \to -\infty} \cot^{-1} t = 0.$$

   b. Let $t = \sin x$, then $t \to 0^+$ as $x \to 0^+$. Thus,

   $$\lim_{x \to 0^+} \ln(\sin x) = \lim_{t \to 0^+} \ln t = -\infty.$$

Exercises X.3 (page 259)

9. Evaluation of the given limits.

   a. The function $\cos(\pi x)$ is continuous everywhere; thus,

   $$\lim_{x \to 1} \cos(\pi x) = \cos(\pi) = -1.$$

   The function $\sin(x-1)$ is positive for $x \to 1^+$ and negative for $x \to 1^-$; thus,

   $$\lim_{x \to 1^+} \sin(x-1) = 0^+ \quad \text{and} \quad \lim_{x \to 1^-} \sin(x-1) = 0^-.$$

   Hence,

   $$\lim_{x \to 1^+} \frac{1}{\sin(x-1)} = \infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{1}{\sin(x-1)} = -\infty.$$

   By Theorem 10.2 (page 251)

   $$\lim_{x \to 1^+} \frac{\cos(\pi x)}{\sin(x-1)} = -\infty \quad \text{and} \quad \lim_{x \to 1^-} \frac{\cos(\pi x)}{\sin(x-1)} = \infty.$$

   Therefore,

   $$\lim_{x \to 1} \frac{\cos(\pi x)}{\sin(x-1)} \quad \text{does not exist.}$$

b. Since $x^2 > 0$ is continuous everywhere and

$$x^3 - 8 = (x-2)(x^2+2x+4) > 0 \quad \text{for } x \to 2^+,$$

we have that

$$\lim_{x\to 2^+} x^2 = 4 \quad \text{and} \quad \lim_{x\to 2^+} (x-2)(x^2+2x+4) = 0^+.$$

By Theorem 10.2 (page 251)

$$\lim_{x\to 2^+} \frac{x^2}{x^3-8} = \infty.$$

c. Since $x^2 > 0$ is continuous everywhere and

$$x^3 - 8 = (x-2)(x^2+2x+4) < 0 \quad \text{for } x \to 2^-,$$

we have that

$$\lim_{x\to 2^-} x^2 = 4 \quad \text{and} \quad \lim_{x\to 2^-} (x-2)(x^2+2x+4) = 0^-.$$

By Theorem 10.2 (page 251)

$$\lim_{x\to 2^+} \frac{x^2}{x^3-8} = -\infty.$$

10. By Exercise 1 above, one vertical asymptotes of the function

$$\frac{\cos(\pi x)}{\sin(x-1)} \quad \text{is } x = 1.$$

We have that $(x-1) \to \pi^+$ as $x \to (\pi-1)^+$; thus, if $t = x - 1$,

$$\lim_{x\to(\pi-1)^+} \sin(x-1) = \lim_{t\to\pi^+} \sin t = 0^-.$$

Hence,

$$\lim_{x\to(\pi-1)^+} \frac{1}{\sin(x-1)} = -\infty.$$

Since $\cos(\pi x)$ is continuous everywhere

$$\lim_{x \to (\pi-1)^+} \cos(\pi x) = \cos(\pi(\pi - 1)) < 0.$$

By Theorem 10.2 (page 251)

$$\lim_{x \to (\pi-1)^+} \frac{\cos(\pi x)}{\sin(x-1)} = \infty.$$

The vertical line $x = \pi - 1$ is another vertical asymptote of the given function.

11. We do not apply Theorem 10.4 (page 257), to evaluate the limit

$$\lim_{x \to 0} \frac{x+1}{\sin\left(\frac{1}{x}\right)} \quad \text{because} \quad \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \quad \text{does not exist.}$$

12. Evaluation of the given limits.

a. If $t = \dfrac{1}{x}$, then $t \to 0^+$ as $x \to \infty$; thus,

$$\lim_{x \to \infty} \ln\left(\frac{1}{x}\right) = \lim_{t \to 0^+} \ln t = -\infty.$$

b. We have that

$$\lim_{x \to -\infty} e^x = 0^+ \quad \Rightarrow \quad \lim_{x \to -\infty} \frac{1}{e^x} = \infty.$$

Also,

$$\lim_{x \to -\infty} 1 - x = \infty.$$

By Theorem 10.4 (page 257)

$$\lim_{x \to -\infty} \frac{1-x}{e^x} = \lim_{x \to -\infty} (1-x)\left(\frac{1}{e^x}\right) = -\infty.$$

13. The application of Theorem 10.2 below is incorrect because

$$\begin{aligned}
\lim_{x\to\pi/2^+} \frac{x-\pi}{\cos x} &= \lim_{x\to\pi/2^+} (x-\pi)\left[\frac{1}{\cos x}\right] \\
&= \lim_{x\to\pi/2^+} (x-\pi)\left[\lim_{x\to\pi/2^+} \frac{1}{\cos x}\right] && \text{the Laws of Limits does n} \\
&= \left(-\frac{\pi}{2}\right)\frac{1}{0} && \text{division by zero is ur} \\
&= \left(-\frac{\pi}{2}\right)(\infty) = -\infty && \text{multiplication by } \infty \text{ is und}
\end{aligned}$$

The correct application is as follows. By continuity

$$\lim_{x\to\pi/2^+} \frac{x-\pi}{=}\frac{\pi}{2}-\pi = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x\to\pi/2^+} \cos x = 0^-.$$

Hence;

$$\lim_{x\to\pi/2^+} \frac{1}{\cos x} = -\infty.$$

By Theorem 10.4 (page 257)

$$\lim_{x\to\pi/2^+} \frac{x-\pi}{\cos x} = \infty.$$

Index